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Local stability for a general wetting functional

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Abstract

We compute the second variation of a general energy functional describing a drop of incompressible liquid in contact with a fluid environment and a rigid substrate structurally inhomogeneous and arbitrarily curved. Both surface and line tensions, residing on the drop's interfaces and along its contact line, contribute to the energy functional. Our method is purely intrinsic, as it does not resort to any representation of the drop's shape. From the energy's second variation we also derive a criterion for the local stability of an equilibrium configuration of the drop.

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1. Introduction

The equilibrium of a liquid droplet laid on a solid substrate has already been studied for more than two centuries: in the absence of external forces, the equilibrium shape of the free surface of the droplet is a surface of constant mean curvature, and the constact angle, that is, the angle that the droplet makes with the substrate, is constant, if the substrate is homogeneous. Its value is determined by the well-known Young–Laplace formula. It is, however, crucial to ascertain whether a given equilibrium shape is stable or not, since only stable, or at least metastable equilibrium configurations can indeed be observed. Studies on the stability of fluid surfaces are rare, especially when constraints are imposed on the energy functional, which must be obeyed up to second order in the perturbations to the admissible shapes. Sekimoto *et al* [1] studied the stability of several wetting morphologies, when a droplet lays on a planar substrate. In particular, they proved analytically that a spherical cap is stable, at least in the limiting case where the free surface of the liquid is nearly parallel to the substrate, that is, the case where the contact angle is small. The authors drew this conclusion from the study of a linear partial differential equation: a spherical drop sitting on a flat substrate is stable against all perturbations that move the contact line, apart from a uniform translation. Another wetting morphology, which in [1] was studied in a special case, is the fluid *ridge*, where a liquid is

drawn along a solid wedge. This geometric setting has recently been studied in more detail by Roy and Schwartz [2] who considered in general a cylindrical droplet laid on a cylindrical substrate. Though the method of their analysis is confined to this two-dimensional case and cannot easily be extended to less symmetric shapes, in [2] the role of the substrate's curvature in the stability of the drop is systematically explored for the first time.

Along a different vein, the analysis of Lenz and Lipowsky [3] obtained a general stability criterion, for arbitrary wetting morphologies, in the presence of a *structured* substrate, that is, a substrate whose chemical properties depend on the position. In this analysis, however, the substrate is flat and no effect of curvature is at play. Yet another approach was pursued by Gelfand and Lipowsky [4] who explored how a *uniformly* curved substrate such as a sphere or a cylinder modifies the surface phase diagram of a fluid droplet lying on it. They incorporated the curvature into an effective bulk order-field entering a Landau model. Using a simplified free energy functional which is tractable analytically, they showed that curvature suppresses pre-wetting transitions.

In the mathematical literature, a clear distinction is made between the concepts of *stability* and *minimality*. Given a functional $\mathcal{F}$ defined on a Banach space $\mathbb{B}$, a point P in $\mathbb{B}$ is *stable* for $\mathcal{F}$ if for all $Q \in \mathbb{B}$ the function $f(\varepsilon) := \mathcal{F}(P + \varepsilon Q)$ attains a minimum at $\varepsilon = 0$. On the other hand, P is a *strict local minimum* for $\mathcal{F}$ in $\mathbb{B}$ if there is $\varepsilon > 0$ such that $\mathcal{F}(P) < \mathcal{F}(Q)$ for all $Q \neq P$ in $\mathbb{B}$ that satisfy the inequality $\|Q - P\|_{\mathbb{B}} < \varepsilon$, where $\|\cdot\|_{\mathbb{B}}$ denotes the norm in $\mathbb{B}$. A strict local minimum is also stable, but a stable point can fail to be a strict local minimum.

A considerable literature has been concerned with the stability of capillary surfaces, mostly in the case where gravity is absent and the substrate supporting the drop is homogeneous (see, in particular, [5, 6]). For the question about strict local minima in capillary theory, the reader is especially referred to a series of papers by Vogel [7–9], where the proposed minimality criteria stem from requiring the second variation of the energy functional to be *strongly* positive, a condition which avoids the difficulty that Finn [10] pointed out in this context by remarking that the mere requirement that the second energy variation be positive cannot guarantee that the energy actually attains a minimum.

The aim of this paper is to prove a stability criterion for wetting, which would apply to structured, inhomogeneous substrates, arbitrarily curved and bearing an adhesion effect possibly diluted in space. This effect is especially relevant to the stability of very small droplets, typically below the micron range, for which an effective interface potential accounts for the balance of the forces between the fluid and the substrate (see, for example, [11]). For drops of this size, a line tension also arises along the line where the drop, the substrate and the environment fluid are in contact; this tension, which is negligible for macroscopic drops, is likely to play a role in the stability of wetting on the sub-micron scale. We assess the stability of the equilibrium shapes of a drop by computing the second variation of a general energy functional. This leads us to a local stability criterion which is especially valuable whenever the equilibrium shape of a drop is not unique. One such circumstance occurs for droplets sitting on a fibre: it is well known that the equilibrium shape of a droplet can be either clamshell-like or barrel-like, depending on the liquid volume, the contact angle and the fibre radius [12, 13].

The paper is organized as follows. In section 2, we present a variational formulation of the wetting problem studied here and we arrive at the corresponding equilibrium equations for an adhering drop. In section 3, we set forth our method to compute the second variation of the general energy functional introduced in section 2: the major technical difficulty addressed here arises from enforcing the constraint on the volume enclosed by the drop, dictated by the incompressibility, and the constraint on the persistence of the contact line on the substrate. Both these geometric constraints must be obeyed up to second order in the perturbations to the equilibrium shape. In section 4, we then arrive at a stability criterion for adhering drops,

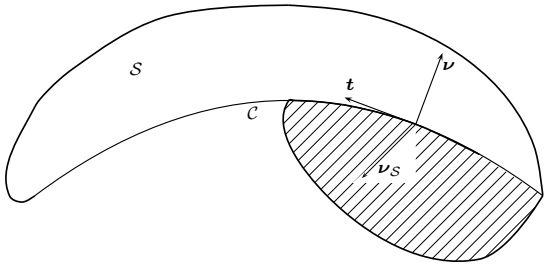

Figure 1. A smooth surface $\mathcal{S}$ and its border $\mathcal{C}$: $\{\boldsymbol{t}, \boldsymbol{\nu}_{\mathcal{S}}, \boldsymbol{\nu}\}$ is the Darboux trihedron.

by computing the least eigenvalue of the second variation appropriately constrained. Finally, in section 5, we draw conclusions for this work and we comment on the significance of the line tension within our stability criterion. The paper concludes with seven appendices, mostly technical in nature. In particular, the second appendix also illustrates by a simple example the meaning of the second-order variation of the drop's shape.

2. Variational formulation

In this section we prepare the way to the stability analysis developed in the following sections. We first collect some preliminary mathematical results to make our development self-consistent, then we posit the energy functional for a wetting drop and finally we arrive at the equilibrium equations for the drop.

2.1. Mathematical preliminaries

Let $\mathcal{S}$ be a smooth orientable surface in the three-dimensional space with border on a smooth closed curve $\mathcal{C}$ (see figure 1). An orientation is assigned to $\mathcal{S}$ by prescribing the unit normal $\boldsymbol{\nu}$; the outer conormal $\boldsymbol{\nu}_{\mathcal{S}}$ is defined on $\mathcal{C}$ as the outward unit vector tangent to $\mathcal{S}$ and orthogonal to $\mathcal{C}$; the border is further oriented so that its unit tangent is $\boldsymbol{t} := \boldsymbol{\nu}_{\mathcal{S}} \times \boldsymbol{\nu}$. The trihedron $\{\boldsymbol{t}, \boldsymbol{\nu}_{\mathcal{S}}, \boldsymbol{\nu}\}$ thus defined along $\mathcal{C}$ is called the Darboux trihedron (see p 261 of [14]). Let s be the arc-length of $\mathcal{C}$ oriented like $\boldsymbol{t}$. The Darboux trihedron varies along $\mathcal{C}$ according to the Darboux equations:

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \kappa_g \boldsymbol{\nu}_{\mathcal{S}} + \kappa_n \boldsymbol{\nu} \\ \dfrac{\mathrm{d}\boldsymbol{\nu}_{\mathcal{S}}}{\mathrm{d}s} = -\kappa_g \boldsymbol{t} - \tau_g \boldsymbol{\nu} \\ \dfrac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s} = -\kappa_n \boldsymbol{t} + \tau_g \boldsymbol{\nu}_{\mathcal{S}} \end{cases} \tag{2.1}$$

where

$$\kappa_n := \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} \cdot \boldsymbol{\nu} \qquad \kappa_g := \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} \cdot \boldsymbol{\nu}_{\mathcal{S}} \qquad \text{and} \quad \tau_g := \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}s} \cdot \boldsymbol{\nu}_{\mathcal{S}} \tag{2.2}$$

are, respectively, the *normal curvature*, the *geodesic curvature* and the *geodesic torsion* of $\mathcal{C}$ relative to $\mathcal{S}$. In the following we extensively employ the surface-divergence theorem (see, for example, p 87 of [15]), which states that a smooth vector field $\boldsymbol{u}$ defined on the surface $\mathcal{S}$ satisfies the equation

$$\int_{\mathcal{S}} \mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}\, \mathrm{d}a = \int_{\mathcal{S}} H u_{\nu}\, \mathrm{d}a + \int_{\mathcal{C}} \boldsymbol{u} \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s \tag{2.3}$$

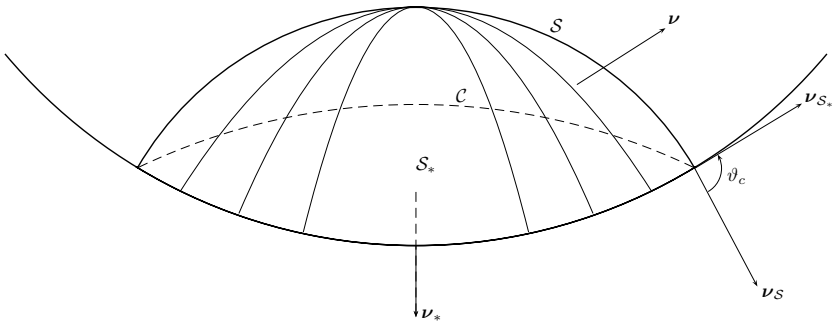

Figure 2. Sketch of a drop deposited on a curved solid substrate. The boundary of the drop is composed of the *free* surface $\mathcal{S}$ and the *adhering* surface $\mathcal{S}_*$. The *contact* line $\mathcal{C}$ is the common border of $\mathcal{S}$ and $\mathcal{S}_*$.

where $\mathrm{div}_{\mathrm{s}}\boldsymbol{u}$ is the *surface divergence* of $\boldsymbol{u}$, that is, the trace of the surface gradient $\nabla_{\mathrm{s}}\boldsymbol{u}$ on $\mathcal{S}$, $u_\nu := \boldsymbol{u}\cdot\boldsymbol{\nu}$ is the normal component of $\boldsymbol{u}$ and H is the total curvature of $\mathcal{S}$. To make precise the sign convention adopted for H, we note that

$$\nabla_{\mathrm{s}}\boldsymbol{\nu} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \tag{2.4}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are tangent unit vectors along the principal directions of $\mathcal{S}$ and σ_1, σ_2 are the corresponding principal curvatures. Here the total curvature of $\mathcal{S}$ is $H := \mathrm{div}_{\mathrm{s}}\boldsymbol{\nu} = \sigma_1 + \sigma_2$, while the Gaussian curvature of $\mathcal{S}$, soon to be employed, is $K := \sigma_1\sigma_2$. Moreover, when $\mathcal{S}$ is sufficiently smooth the normal field $\boldsymbol{\nu}$ obeys [16]

$$\Delta_{\mathrm{s}}\boldsymbol{\nu} = \nabla_{\mathrm{s}} H + (2K - H^2)\boldsymbol{\nu} \tag{2.5}$$

where Δ_{s} denotes the surface Laplacian; in particular, $\Delta_{\mathrm{s}}\boldsymbol{\nu} := \mathrm{div}_{\mathrm{s}}\nabla_{\mathrm{s}}\boldsymbol{\nu}$. Similarly, one shows that [16]

Lemma 2.1. *The tensor*

$$K\mathbf{I} - H(\nabla_{\mathrm{s}}\boldsymbol{\nu}) + (\nabla_{\mathrm{s}}\boldsymbol{\nu})^2 \tag{2.6}$$

where $\mathbf{I}$ *is the identity, vanishes on all vectors tangent to* $\mathcal{S}$.

A particular result that will soon be employed is stated in the following lemma; its proof is given in appendix A.

Lemma 2.2. *Let* $\boldsymbol{g}$ *be a regular tangent vector field defined on* $\mathcal{S}$. *Then*

$$\int_{\mathcal{S}} I_2(\nabla_{\mathrm{s}}\boldsymbol{g})\,\mathrm{d}a = \frac{1}{2}\int_{\mathcal{S}} K|\boldsymbol{g}|^2\,\mathrm{d}a - \frac{1}{2}\int_{\mathcal{C}}[(\nabla_{\mathrm{s}}\boldsymbol{g})\boldsymbol{g} - (\mathrm{div}_{\mathrm{s}}\boldsymbol{g})\boldsymbol{g}]\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s \tag{2.7}$$

where, for any second-order tensor $\mathbf{L}$, $I_2(\mathbf{L})$ *is the second invariant defined as*

$$I_2(\mathbf{L}) := \tfrac{1}{2}[(\mathrm{tr}\,\mathbf{L})^2 - \mathrm{tr}(\mathbf{L}^2)]. \tag{2.8}$$

2.2. Free-energy functional

We consider a liquid drop of prescribed volume deposited on a curved, adhesive substrate. We call $\mathcal{B}$ the region in space occupied by the drop and $\partial\mathcal{B}$ its whole boundary, which is composed of the *free* surface $\mathcal{S}$, where the drop is in contact with the environment fluid, and the *adhering* surface $\mathcal{S}_*$, where the drop is in contact with the substrate. The common border of $\mathcal{S}$ and $\mathcal{S}_*$ is the *contact* line $\mathcal{C}$ of the drop, where three distinct phases meet (see figure 2). The free-energy

functional $\mathcal{F}$ of the drop is composed of four different terms, which are separately described below:

$$\mathcal{F}[\mathcal{B}] = \mathcal{F}_b[\mathcal{B}] + \mathcal{F}_s[\mathcal{S}] + \mathcal{F}_a[\mathcal{S}_*] + \mathcal{F}_\ell[\mathcal{C}]. \tag{2.9}$$

The first term is the bulk energy

$$\mathcal{F}_b[\mathcal{B}] := \int_{\mathcal{B}} f \, \mathrm{d}v \tag{2.10}$$

which models the action of an external field on the whole drop. The scalar function f generally depends on the position in space; it would represent not only gravity, but also any diluted interaction between the drop and the substrate, in the spirit of [11, 17]. The second term is the interfacial energy of the free surface $\mathcal{S}$:

$$\mathcal{F}_s[\mathcal{S}] := \gamma \int_{\mathcal{S}} \mathrm{d}a \tag{2.11}$$

where γ is the constant surface tension of the drop. The next term,

$$\mathcal{F}_a[\mathcal{S}_*] := \int_{\mathcal{S}_*} (\gamma - w) \, \mathrm{d}a \tag{2.12}$$

describes the adhesive properties of the substrate through the *adhesion* potential w. Here the interfacial energy on the substrate is conventionally written as $\gamma - w$, implying that $w > 0$ for an adhesive substrate. Finally,

$$\mathcal{F}_\ell[\mathcal{C}] := \int_{\mathcal{C}} \tau \, \mathrm{d}s \tag{2.13}$$

is the free energy of the contact line $\mathcal{C}$, where τ can be interpreted as a line tension.

Here both w and τ are taken as functions of the position on the substrate to describe constitutive material inhomogeneities. When both w and τ are constant, arbitrary geometric microstructures are still possible in the substrate. Assuming that both the adhesion potential and the line tension are variable on the substrate makes the free-energy functional (2.9) very general and our stability analysis accordingly more delicate.

2.3. Shape variation

The functional $\mathcal{F}$ is subject to the constraint on the volume of $\mathcal{B}$ and to the condition that $\mathcal{S}_*$ be part of the substrate. All admissible variations of $\mathcal{F}$ must preserve the drop volume and ensure that $\mathcal{S}_*$ *glides* on the substrate. In particular, a correct stability analysis relies on enforcing both these requirements up to second order. Formally, we perturb the shape $\mathcal{B}$ of the drop by mapping every point p into p_ε:

$$p \mapsto p_\varepsilon := p + \varepsilon \boldsymbol{u} + \varepsilon^2 \boldsymbol{v} \tag{2.14}$$

where ε is a perturbation parameter, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are smooth vector fields describing the first- and second-order variations of $\mathcal{B}$, respectively. The meaning of the second-order variation $\boldsymbol{v}$ is illustrated in appendix B. A field like this was similarly introduced by Peterson in a completely different context (see [18, 19]). Here we make explicit the constraints that must be obeyed by both $\boldsymbol{u}$ and $\boldsymbol{v}$. To this end, we first describe how the shape variation in (2.14) affects volumes, areas and lengths in general.

Let $\mathcal{B}_\varepsilon$ be the image of $\mathcal{B}$ under the mapping $p \mapsto p_\varepsilon$ in (2.14) and let V_ε be the volume of $\mathcal{B}_\varepsilon$ and A_ε the area of $\partial\mathcal{B}_\varepsilon$. It is known that

$$V_\varepsilon = \int_{\mathcal{B}_\varepsilon} \det \mathbf{F}_\varepsilon \, \mathrm{d}v \tag{2.15}$$

where

$$\mathbf{F}_\varepsilon := \mathbf{I} + \varepsilon\nabla\boldsymbol{u} + \varepsilon^2\nabla\boldsymbol{v}$$

and that

$$A_\varepsilon = \int_{\partial\mathcal{B}} |\mathbf{F}_\varepsilon^*\boldsymbol{\nu}|\,\mathrm{d}a$$

where $\mathbf{F}_\varepsilon^* := \det\mathbf{F}_\varepsilon(\mathbf{F}_\varepsilon^{-1})^{\mathsf{T}}$ is the *adjugate* tensor of $\mathbf{F}_\varepsilon$ (see pp 53–54 of [20]). Here and in the following a superscript $^{\mathsf{T}}$ denotes tensor transposition. If we set $\boldsymbol{w} := \boldsymbol{u} + \varepsilon\boldsymbol{v}$, then

$$\det\mathbf{F}_\varepsilon = \det(\mathbf{I} + \varepsilon\boldsymbol{w}) = 1 + \varepsilon\,\mathrm{div}\,\boldsymbol{w} + \varepsilon^2 I_2(\nabla\boldsymbol{w}) + O(\varepsilon^3) \tag{2.16}$$

and it follows from (2.15) that

$$V_\varepsilon = \int_{\mathcal{B}} \det\mathbf{F}_\varepsilon\,\mathrm{d}v = \int_{\mathcal{B}} [1 + \varepsilon\,\mathrm{div}\,\boldsymbol{w} + \varepsilon^2 I_2(\nabla\boldsymbol{u})]\,\mathrm{d}v + O(\varepsilon^3). \tag{2.17}$$

As proved in section 3.8 of [15], the following identity holds for any regular vector field $\boldsymbol{h}$ defined on $\mathcal{B}$:

$$\mathrm{tr}[(\nabla\boldsymbol{h})^2] - (\mathrm{div}\,\boldsymbol{h})^2 = \mathrm{div}[(\nabla\boldsymbol{h})\boldsymbol{h} - (\mathrm{div}\,\boldsymbol{h})\boldsymbol{h}]. \tag{2.18}$$

Moreover, it can also be proved (see p 160 of [15]) that

$$(\nabla\boldsymbol{h})\boldsymbol{h} - (\mathrm{div}\,\boldsymbol{h})\boldsymbol{h} = (\nabla_{\mathrm{s}}\boldsymbol{h})\boldsymbol{h} - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{h})\boldsymbol{h}. \tag{2.19}$$

Hence, by (2.8), (2.18) and (2.19), also using the divergence theorem and recalling the definition of $\boldsymbol{w}$, we arrive at

$$V_\varepsilon = V + \varepsilon\int_{\partial\mathcal{B}} u_\nu\,\mathrm{d}a + \varepsilon^2\int_{\partial\mathcal{B}} v_\nu\,\mathrm{d}a + \frac{1}{2}\varepsilon^2\int_{\partial\mathcal{B}} [(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})u_\nu - \boldsymbol{u}\cdot\boldsymbol{a}]\,\mathrm{d}a + O(\varepsilon^3) \tag{2.20}$$

where V is the volume of $\mathcal{B}$, $v_\nu := \boldsymbol{v}\cdot\boldsymbol{\nu}$ and

$$\boldsymbol{a} := (\nabla_{\mathrm{s}}\boldsymbol{u})^{\mathsf{T}}\boldsymbol{\nu} \tag{2.21}$$

is a field everywhere tangent to $\partial\mathcal{B}$. Similarly, since

$$\mathbf{F}_\varepsilon^{-1} = \mathbf{I} - \varepsilon(\nabla\boldsymbol{w})^{\mathsf{T}} + \varepsilon^2[(\nabla\boldsymbol{w})^{\mathsf{T}}]^2 + O(\varepsilon^3)$$

by the identities

$$(\mathrm{div}\,\boldsymbol{w})\boldsymbol{\nu} - (\nabla\boldsymbol{w})^{\mathsf{T}}\boldsymbol{\nu} = (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{w})\boldsymbol{\nu} - (\nabla_{\mathrm{s}}\boldsymbol{w})^{\mathsf{T}}\boldsymbol{\nu}$$
$$[((\nabla\boldsymbol{w})^{\mathsf{T}})^2 + I_2(\nabla\boldsymbol{w})\mathbf{I} - (\mathrm{div}\,\boldsymbol{w})(\nabla\boldsymbol{w})^{\mathsf{T}}]\boldsymbol{\nu} = [((\nabla_{\mathrm{s}}\boldsymbol{w})^{\mathsf{T}})^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{w})\mathbf{I} - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{w})(\nabla_{\mathrm{s}}\boldsymbol{w})^{\mathsf{T}}]\boldsymbol{\nu}$$

one easily arrives at the following expression for the local surface dilation factor:

$$|\mathbf{F}_\varepsilon^*\boldsymbol{\nu}| = 1 + \varepsilon\,\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u} + \varepsilon^2\big[\mathrm{div}_{\mathrm{s}}\,\boldsymbol{v} + \tfrac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\big] + O(\varepsilon^3) \tag{2.22}$$

where (2.21) and the definition of $\boldsymbol{w}$ have also been used. Thus we conclude that

$$A_\varepsilon = A + \varepsilon\int_{\partial\mathcal{B}} \mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}\,\mathrm{d}a + \varepsilon^2\int_{\partial\mathcal{B}}\left[\mathrm{div}_{\mathrm{s}}\,\boldsymbol{v} + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\right]\mathrm{d}a + O(\varepsilon^3) \tag{2.23}$$

where A is the area of $\partial\mathcal{B}$.

Finally, (2.14) also induces a deformation of the contour $\mathcal{C}$. If s is the arc-length on $\mathcal{C}$, then

$$\frac{\mathrm{d}p_\varepsilon}{\mathrm{d}s} = \boldsymbol{t} + \varepsilon\boldsymbol{u}' + \varepsilon^2\boldsymbol{v}' + O(\varepsilon^3)$$

where a prime denotes differentiation with respect to s. Thus, the local dilation factor along $\mathcal{C}$ is

$$\left|\frac{\mathrm{d}p_\varepsilon}{\mathrm{d}s}\right| = 1 + \varepsilon \boldsymbol{u}' \cdot \boldsymbol{t} + \frac{1}{2}\varepsilon^2[2\boldsymbol{t} \cdot \boldsymbol{v}' + \boldsymbol{u}' \cdot \boldsymbol{u}' - (\boldsymbol{u}' \cdot \boldsymbol{t})^2] + O(\varepsilon^3). \tag{2.24}$$

We call $\boldsymbol{\nu}_*$ and $\boldsymbol{\nu}_{\mathcal{S}_*}$ the outer unit normal to $\mathcal{B}$ on $\mathcal{S}_*$ and the outer conormal to $\mathcal{S}_*$ along the border $\mathcal{C}$. In the following, the angle ϑ_c made by $\boldsymbol{\nu}_\mathcal{S}$ and $\boldsymbol{\nu}_{\mathcal{S}_*}$ is often referred to as the *contact angle* (see figure 2). Let $\boldsymbol{t}_* = -\boldsymbol{t}$. Then $\{\boldsymbol{t}_*, \boldsymbol{\nu}_{\mathcal{S}_*}, \boldsymbol{\nu}_*\}$ is the Darboux trihedron on $\mathcal{C}$ relative to $\mathcal{S}_*$ oriented precisely like the Darboux trihedron $\{\boldsymbol{t}, \boldsymbol{\nu}_\mathcal{S}, \boldsymbol{\nu}\}$ on $\mathcal{C}$ relative to $\mathcal{S}$. In particular, $\boldsymbol{\nu}_\mathcal{S}$ can be represented as

$$\boldsymbol{\nu}_\mathcal{S} = \cos\vartheta_c \boldsymbol{\nu}_{\mathcal{S}_*} + \sin\vartheta_c \boldsymbol{\nu}_*. \tag{2.25}$$

In complete analogy with (2.1), the Darboux equations obeyed by $\{\boldsymbol{t}_*, \boldsymbol{\nu}_{\mathcal{S}_*}, \boldsymbol{\nu}_*\}$ are

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{t}_*}{\mathrm{d}s_*} = \kappa_g^* \boldsymbol{\nu}_{\mathcal{S}_*} + \kappa_n^* \boldsymbol{\nu}_* \\ \dfrac{\mathrm{d}\boldsymbol{\nu}_{\mathcal{S}_*}}{\mathrm{d}S_*} = -\kappa_g^* \boldsymbol{t}_* - \tau_g^* \boldsymbol{\nu}_* \\ \dfrac{\mathrm{d}\boldsymbol{\nu}_*}{\mathrm{d}s_*} = -\kappa_n^* \boldsymbol{t}_* + \tau_g^* \boldsymbol{\nu}_{\mathcal{S}_*} \end{cases} \tag{2.26}$$

where $s_* = -s$ and κ_n^*, κ_g^* and τ_g^* are defined by the analogues of (2.2).

The fields $\boldsymbol{u}$ and $\boldsymbol{v}$ are not free on the adhering surface $\mathcal{S}_*$ but they must ensure that the contact between the drop and the substrate is preserved by the mapping (2.14), up to second order in ε. As explained in appendix B, this requires that

$$\boldsymbol{u} \cdot \boldsymbol{\nu}_* = 0 \qquad \text{and} \qquad \boldsymbol{v} \cdot \boldsymbol{\nu}_* = -\tfrac{1}{2}\boldsymbol{u} \cdot (\nabla_\mathrm{s}\boldsymbol{\nu}_*)\boldsymbol{u} \quad \text{on } \mathcal{S}_*. \tag{2.27}$$

Thus, only the tangential components of both $\boldsymbol{u}$ and $\boldsymbol{v}$ can be freely chosen on $\mathcal{S}_*$. It follows from (2.20) and (2.27) that when the surface $\mathcal{S}_*$ is kept by (2.14) on the substrate, it does not contribute to the variation of the volume enclosed by the drop. In fact, by $(2.27)_1$ and the symmetry of $\nabla_\mathrm{s}\boldsymbol{\nu}_*$, we have that

$$\boldsymbol{a} = (\nabla_\mathrm{s}\boldsymbol{u})^\mathsf{T}\boldsymbol{\nu}_* = -(\nabla_\mathrm{s}\boldsymbol{\nu}_*)\boldsymbol{u} \qquad \text{on } \mathcal{S}_* \tag{2.28}$$

and so, by $(2.27)_2$, (2.20) becomes

$$V_\varepsilon = V + \varepsilon \int_\mathcal{S} u_\nu \,\mathrm{d}a + \varepsilon^2 \int_\mathcal{S} v_\nu \,\mathrm{d}a + \frac{1}{2}\varepsilon^2 \int_\mathcal{S} [(\mathrm{div}_\mathrm{s}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_\mathrm{s}\boldsymbol{u})\boldsymbol{u}] \cdot \boldsymbol{\nu}\,\mathrm{d}a.$$

Hence, enforcing the constraint on the enclosed volume up to second order in ε requires that

$$\int_\mathcal{S} u_\nu \,\mathrm{d}a = 0 \qquad \text{and} \qquad \int_\mathcal{S} (u_\nu \,\mathrm{div}_\mathrm{s}\,\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{a} + 2v_\nu)\,\mathrm{d}a = 0. \tag{2.29}$$

A similar conclusion can also be drawn for the contribution of $\mathcal{S}_*$ to the variation of the total area A of both drop interfaces. Each integral in (2.23) can be split into an integral over $\mathcal{S}$ and another integral over $\mathcal{S}_*$. In particular, by the surface-divergence theorem (2.3) applied to both surfaces and by $(2.27)_1$,

$$\int_\mathcal{S} \mathrm{div}_\mathrm{s}\,\boldsymbol{u}\,\mathrm{d}a = \int_\mathcal{S} H u_\nu \,\mathrm{d}a + \int_\mathcal{C} \boldsymbol{u} \cdot \boldsymbol{\nu}_\mathcal{S}\,\mathrm{d}s \tag{2.30}$$

and

$$\int_{\mathcal{S}_*} \mathrm{div}_\mathrm{s}\,\boldsymbol{u}\,\mathrm{d}a = \int_\mathcal{C} \boldsymbol{u} \cdot \boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s \tag{2.31}$$

while, by $(2.27)_2$,

$$\int_{\mathcal{S}_*} \operatorname{div}_{\mathrm{s}} \boldsymbol{v}\, \mathrm{d}a = -\frac{1}{2}\int_{\mathcal{S}_*} H\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}\, \mathrm{d}a + \int_{\mathcal{C}} \boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\, \mathrm{d}s. \tag{2.32}$$

By using equations (2.28)–(2.32) and resorting to both lemmas 2.1 and 2.2, we finally reduce (2.23) to

$$A_\varepsilon = A + \varepsilon\left(\int_{\mathcal{S}} Hu_\nu\, \mathrm{d}a + \int_{\mathcal{C}} \boldsymbol{u}\cdot(\boldsymbol{\nu}_{\mathcal{S}} + \boldsymbol{\nu}_{\mathcal{S}_*})\, \mathrm{d}s\right) + \varepsilon^2\left(\int_{\mathcal{S}}\left[Hv_\nu + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\right]\mathrm{d}a + \int_{\mathcal{C}}\left\{\boldsymbol{v}\cdot(\boldsymbol{\nu}_{\mathcal{S}} + \boldsymbol{\nu}_{\mathcal{S}_*}) + \frac{1}{2}[(\operatorname{div}_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} - (\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\right\}\mathrm{d}s\right) + O(\varepsilon^3). \tag{2.33}$$

It is worth noting that equation (2.33), when applied only for the area A_* of the single surface $\mathcal{S}_*$, becomes

$$A_{*\varepsilon} = A_* + \varepsilon\int_{\mathcal{C}} \boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\, \mathrm{d}s + \varepsilon^2\int_{\mathcal{C}}\left\{\boldsymbol{v} + \frac{1}{2}((\operatorname{div}_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} - (\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u})\right\}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\, \mathrm{d}s + O(\varepsilon^3). \tag{2.34}$$

2.4. Energy variations

To compute the first and the second variations of $\mathcal{F}$, it is expedient to expand the functions f, w and τ as follows:

$$\begin{cases} f(p_\varepsilon) = f(p) + \varepsilon\nabla f\cdot\boldsymbol{u} + \frac{1}{2}\varepsilon^2[(\nabla^2 f)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla f\cdot\boldsymbol{v}] + O(\varepsilon^3) \\ w(p_\varepsilon) = w(p) + \varepsilon\nabla_{\mathrm{s}} w\cdot\boldsymbol{u} + \frac{1}{2}\varepsilon^2\left[\left(\nabla_{\mathrm{s}}^2 w\right)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla_{\mathrm{s}} w\cdot\boldsymbol{v}\right] + O(\varepsilon^3) \\ \tau(p_\varepsilon) = \tau(p) + \varepsilon\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{u} + \frac{1}{2}\varepsilon^2\left[\left(\nabla_{\mathrm{s}}^2\tau\right)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{v}\right] + O(\varepsilon^3). \end{cases} \tag{2.35}$$

Note that, while f is defined in the whole $\mathcal{B}$, both w and τ are defined only on $\mathcal{S}_*$.

It follows from $(2.35)_1$ and (2.16) that

$$\mathcal{F}_b[\mathcal{B}_\varepsilon] := \mathcal{F}_b[\mathcal{B}] + \varepsilon\int_{\mathcal{B}}(f(p)\operatorname{div}\boldsymbol{u} + \nabla f\cdot\boldsymbol{u})\, \mathrm{d}v + \varepsilon^2\int_{\mathcal{B}}\left\{f\operatorname{div}\left[\boldsymbol{v} + \frac{1}{2}(\operatorname{div}\boldsymbol{u})\boldsymbol{u} - \frac{1}{2}(\nabla\boldsymbol{u})\boldsymbol{u}\right] + (\nabla f\cdot\boldsymbol{u})(\operatorname{div}\boldsymbol{u}) + \frac{1}{2}[(\nabla^2 f)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla f\cdot\boldsymbol{v}]\right\}\mathrm{d}v$$

where the identity (2.18) has also been employed. Moreover, using the identities

$$\operatorname{div}(f\boldsymbol{u}) = f\operatorname{div}\boldsymbol{u} + \nabla f\cdot\boldsymbol{u}$$
$$\operatorname{div}[(\nabla f\cdot\boldsymbol{u})\boldsymbol{u}] = (\nabla f\cdot\boldsymbol{u})\operatorname{div}\boldsymbol{u} + \boldsymbol{u}\cdot(\nabla\boldsymbol{u})^{\mathsf{T}}(\nabla f) + \boldsymbol{u}\cdot(\nabla^2 f)\boldsymbol{u}$$

leads us to

$$\mathcal{F}_b[\mathcal{B}_\varepsilon] = \mathcal{F}_b[\mathcal{B}] + \varepsilon\int_{\mathcal{B}}\operatorname{div}(f\boldsymbol{u})\, \mathrm{d}v + \varepsilon^2\int_{\mathcal{B}}\operatorname{div}\left\{f\left[\boldsymbol{v} + \frac{1}{2}(\operatorname{div}\boldsymbol{u})\boldsymbol{u} - \frac{1}{2}(\nabla\boldsymbol{u})\boldsymbol{u}\right] + \frac{1}{2}[(\nabla f)\cdot\boldsymbol{u}]\boldsymbol{u}\right\}\mathrm{d}v$$

whence, by resorting to (2.19) and to the divergence theorem, we arrive at

$$\mathcal{F}_b[\mathcal{B}_\varepsilon] = \mathcal{F}_b[\mathcal{B}] + \varepsilon\int_{\partial\mathcal{B}} fu_\nu\, \mathrm{d}a + \varepsilon^2\int_{\partial\mathcal{B}}\left\{f\left[\boldsymbol{v} + \frac{1}{2}(\operatorname{div}_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} - \frac{1}{2}(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}\right]\cdot\boldsymbol{\nu} + \frac{1}{2}[(\nabla f)\cdot\boldsymbol{u}]u_\nu\right\}\mathrm{d}a + O(\varepsilon^3).$$

Then, splitting $\partial\mathcal{B}$ into the union of $\mathcal{S}$ and $\mathcal{S}_*$, also using (2.21), yields

$$\mathcal{F}_b[\mathcal{B}_\varepsilon] = \mathcal{F}_b[\mathcal{B}] + \varepsilon \int_{\mathcal{S}} f u_\nu \,\mathrm{d}a + \varepsilon^2 \int_{\mathcal{S}} \left\{ f \left[\boldsymbol{v} + \frac{1}{2}(\mathrm{div}_{\mathrm{s}}\, \boldsymbol{u})\boldsymbol{u} - \frac{1}{2}(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} \right] \cdot \boldsymbol{\nu} + \frac{1}{2}[(\nabla f)\cdot \boldsymbol{u}]u_\nu \right\} \mathrm{d}a + O(\varepsilon^3) \tag{2.36}$$

where again no contribution arises from the adhering surface $\mathcal{S}_*$.

The sum of the energies $\mathcal{F}_s[\mathcal{S}]$ and $\mathcal{F}_a[\mathcal{S}_*]$ can also be written as

$$\mathcal{F}_s[\mathcal{S}] + \mathcal{F}_a[\mathcal{S}_*] = \gamma A + \mathcal{F}_w[\mathcal{S}_*] \tag{2.37}$$

where

$$\mathcal{F}_w[\mathcal{S}_*] := -\int_{\mathcal{S}_*} w \,\mathrm{d}a. \tag{2.38}$$

It follows from (2.22) and $(2.35)_2$ that

$$\mathcal{F}_w[\mathcal{S}_{*\varepsilon}] := -\int_{\mathcal{S}_*} \left\{ w + \varepsilon\, \mathrm{div}_{\mathrm{s}}(w\boldsymbol{u}) + \varepsilon^2 \left[\mathrm{div}_{\mathrm{s}}(w\boldsymbol{v}) + w\left(\frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\right) + \frac{1}{2}\big(\nabla_{\mathrm{s}}^2 w\big)\boldsymbol{u}\cdot\boldsymbol{u} + \mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}(\nabla_{\mathrm{s}} w)\cdot\boldsymbol{u} \right] \right\} \mathrm{d}a + O(\varepsilon^3) \tag{2.39}$$

where $\mathcal{S}_{*\varepsilon}$ is the image of $\mathcal{S}_*$ under the mapping (2.14). Since, by $(2.27)_1$, the vector field $\boldsymbol{u}$ is purely tangential on $\mathcal{S}_*$, the surface-divergence theorem (2.3) gives

$$\int_{\mathcal{S}_*} \mathrm{div}_{\mathrm{s}}(w\boldsymbol{u})\,\mathrm{d}a = \int_{\mathcal{C}} w\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s$$

and, by $(2.27)_2$, we have that

$$\int_{\mathcal{S}_*} \mathrm{div}_{\mathrm{s}}(w\boldsymbol{v})\,\mathrm{d}a = -\frac{1}{2}\int_{\mathcal{S}_*} Hw\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{C}} w\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s.$$

Thus, we arrive at

$$\mathcal{F}_w[\mathcal{S}_{*\varepsilon}] = -\int_{\mathcal{S}_*} \left\{ w + \varepsilon^2 \left[-\frac{1}{2}Hw\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u} + w\left(\frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\right) + \frac{1}{2}\big(\nabla_{\mathrm{s}}^2 w\big)\boldsymbol{u}\cdot\boldsymbol{u} + (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})(\nabla_{\mathrm{s}} w)\cdot\boldsymbol{u} \right] \right\} \mathrm{d}a - \varepsilon\int_{\mathcal{C}} w\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s - \varepsilon^2\int_{\mathcal{C}} w\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s + O(\varepsilon^3). \tag{2.40}$$

Finally, using (2.24) and $(2.35)_3$, we obtain from (2.13) that

$$\mathcal{F}_\ell[\mathcal{C}_\varepsilon] = \int_{\mathcal{C}} \left\{ 1 + \varepsilon(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{u} + \tau\boldsymbol{t}\cdot\boldsymbol{u}') + \varepsilon^2 \left[\frac{\tau}{2}(2\boldsymbol{t}\cdot\boldsymbol{v}' + \boldsymbol{u}'\cdot\boldsymbol{u}' - (\boldsymbol{u}'\cdot\boldsymbol{t})^2) + \frac{1}{2}\big[\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{v}\big] + (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{u})(\boldsymbol{u}'\cdot\boldsymbol{t}) \right] \right\} \mathrm{d}s + O(\varepsilon^3) \tag{2.41}$$

where $\mathcal{C}_\varepsilon$ is the deformed contact line $\mathcal{C}$. Since

$$\boldsymbol{t}' = (\nabla_{\mathrm{s}}\boldsymbol{t})\boldsymbol{t} \qquad \text{and} \qquad \nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t} = \tau' \tag{2.42}$$

an integration by parts along $\mathcal{C}$, which is a closed contour, transforms (2.41) into

$$\mathcal{F}_\ell[\mathcal{C}_\varepsilon] = \int_{\mathcal{C}} \left\{ 1 + \varepsilon[(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})(\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}) - \tau\kappa_g^*(\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*})] + \varepsilon^2 \left[(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})(\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{v}) - \tau\boldsymbol{v}\cdot(\kappa_g^*\boldsymbol{\nu}_{\mathcal{S}_*} + \kappa_n^*\boldsymbol{\nu}_*) + \frac{\tau}{2}(\boldsymbol{u}'\cdot\boldsymbol{u}' - (\boldsymbol{u}'\cdot\boldsymbol{t})^2) + \frac{1}{2}\big[\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{u}\cdot\boldsymbol{u} + 2\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{v}\big] + (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{u})(\boldsymbol{u}'\cdot\boldsymbol{t}) \right] \right\} \mathrm{d}s + O(\varepsilon^3) \tag{2.43}$$

where both equations $(2.26)_{1,2}$ have been used to write

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{t}_*}{\mathrm{d}s_*} = \kappa_g^*\boldsymbol{\nu}_{\mathcal{S}_*} + \kappa_n^*\boldsymbol{\nu}_*$$

and the identity $\boldsymbol{t}_* = -\boldsymbol{t}$ has been used to see that

$$\nabla_{\mathrm{s}}\tau - \boldsymbol{t}\cdot(\nabla_{\mathrm{s}}\tau)\boldsymbol{t} = (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})\boldsymbol{\nu}_{\mathcal{S}_*}.$$

2.5. Equilibrium equations

We are now in a position to compute the first variation of $\mathcal{F}$ in (2.9), defined as

$$\delta\mathcal{F} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}[\mathcal{B}_\varepsilon]\bigg|_{\varepsilon=0}.$$

The equilibrium equations for $\mathcal{F}$ guarantee that $\delta\mathcal{F}$ vanishes identically for all possible fields $\boldsymbol{u}$. By collecting all first-order contributions in ε from equations (2.33), (2.36), (2.40) and (2.43), also with the aid of (2.20) and (2.37), we readily arrive at the following equilibrium equations:

$$\begin{cases} \gamma H + f = \lambda & \text{on } \mathcal{S} \\ \gamma\cos\vartheta_c + \gamma - w + \nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \tau\kappa_g^* = 0 & \text{along } \mathcal{C} \end{cases} \tag{2.44}$$

where λ is the Lagrange multiplier associated with the constraint on the volume enclosed by the drop. Equation $(2.44)_1$ can be interpreted as a balance of forces across the free surface $\mathcal{S}$ of the drop; λ then represents the pressure difference on the two sides of $\mathcal{S}$. When $f = 0$, since both γ and λ are constant, $(2.44)_1$ prescribes $\mathcal{S}$ to have constant mean curvature $\frac{1}{2}H$. Likewise, when $\tau = 0$ and w is constant, $(2.44)_2$ prescribes the contact angle ϑ_c of the drop to be constant along $\mathcal{C}$. Equation $(2.44)_1$ is the standard Laplace equation, whereas $(2.44)_2$ is the generalized Young equation recently also found by Swain and Lipowsky [21].

3. Second variation

In this section we compute the second variation of the free-energy functional $\mathcal{F}$ in (2.9). The method we employ does not resort to any explicit representation of the surface that supports the drop: this perhaps makes our development formally more involved, but the outcome is more general. We shall prove the following formula:

$$\begin{aligned}\delta^2\mathcal{F} := \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{F}[\mathcal{B}_\varepsilon]\bigg|_{\varepsilon=0} &= \gamma\int_{\mathcal{S}}\left\{|\nabla_{\mathrm{s}}u_\nu|^2 + (2K - H^2 + \partial_\nu f)u_\nu^2\right\}\mathrm{d}a \\ &\quad - \gamma\int_{\mathcal{C}}\left(\frac{H^*}{\sin\vartheta_c} + \cot\vartheta_c H + \kappa_g\right)u_\nu^2\,\mathrm{d}s + \int_{\mathcal{C}}\Big\{\tau(u_{s*}')^2 - \Big[\tau(K^* + (\kappa_g^*)^2) \\ &\quad + \nabla_{\mathrm{s}}w\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \left(\nabla_{\mathrm{s}}^2\tau\right)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\Big]u_{s*}^2\Big\}\,\mathrm{d}s.\end{aligned} \tag{3.1}$$

In equation (3.1), u_ν is subject to $(2.29)_1$ and it is related to $u_{s*} := \boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}$ on $\mathcal{C}$ through

$$u_\nu = \sin\vartheta_c u_{s*}. \tag{3.2}$$

Moreover, $\partial_\nu f := \nabla f\cdot\boldsymbol{\nu}$ is the normal derivative of f on $\mathcal{S}$, H^* and K^* are the total and Gaussian curvatures of $\mathcal{S}_*$, respectively. Clearly, when w is constant and τ vanishes on the substrate, the second line integral in (3.1) vanishes and $\delta^2\mathcal{F}$ acquires a much simpler form, which was proved in [5] and had already been employed in both [6] and [22]. In general, it is remarkable that equation (3.1) depends only on the normal component of $\boldsymbol{u}$ on $\mathcal{S}$, while both

$\boldsymbol{u}$ and $\boldsymbol{v}$ have non-vanishing tangential components. For the ease of the reader, we split below the meandering avenue that led us to (3.1) into several elementary strides.

Henceforth, it is expedient to decompose $\boldsymbol{u}$ on $\mathcal{S}$ as

$$\boldsymbol{u} = \boldsymbol{u}_{\parallel} + u_{\nu}\boldsymbol{\nu} \tag{3.3}$$

where $\boldsymbol{u}_{\parallel}$ is the tangential component of $\boldsymbol{u}$. It is shown in appendix C how to arrive at the following formula:

$$\begin{aligned}\frac{1}{2}(\delta^2\mathcal{F}_b + \gamma\delta^2 A) = {} & \frac{1}{2}\int_{\mathcal{S}} \left\{\gamma\left[|\nabla_{\mathrm{s}} u_{\nu}|^2 + (2K - H^2)u_{\nu}^2\right] + (\partial_{\nu} f)u_{\nu}^2\right\} \mathrm{d}a \\ & + \frac{\gamma}{2}\int_{\mathcal{C}} H u_{\nu}\boldsymbol{u}_{\parallel}\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s - \gamma\int_{\mathcal{C}} u_{\nu}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel}\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s \\ & - \frac{\gamma}{2}\int_{\mathcal{C}} [(\nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel} - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel}]\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s + \gamma\int_{\mathcal{C}}(1+\cos\vartheta_c)\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s \\ & - \frac{\gamma}{2}\int_{\mathcal{C}} [(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s - \frac{\gamma}{2}\int_{\mathcal{C}}\sin\vartheta_c\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}\,\mathrm{d}s\end{aligned} \tag{3.4}$$

which is the first building block of our development.

By (2.37), another building block of our computation is the second variation of the functional $\mathcal{F}_w$ defined in (2.38) (see appendix D):

$$\frac{1}{2}\delta^2\mathcal{F}_w[\mathcal{S}_*] = -\int_{\mathcal{C}}\left\{w\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \frac{w}{2}[(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \frac{1}{2}(\nabla_{\mathrm{s}}w)\cdot\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*})\right\}\mathrm{d}s. \tag{3.5}$$

We now write the second variation of $\mathcal{F}_\ell$ in (2.13):

$$\begin{aligned}\frac{1}{2}\delta^2\mathcal{F}_\ell[\mathcal{C}] = {} & \int_{\mathcal{C}}\left\{(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})(\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{v}) - \tau\left[\kappa_g^*\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \frac{1}{2}\kappa_n^*\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}\right]\right. \\ & + \frac{1}{2}\tau[(u'_{s*} - u_{t*}\kappa_g^*)^2 + (u_{s*}\tau_g^* - u_{t*}\kappa_n^*)^2] - \frac{1}{2}u_{t*}^2\kappa_g^*(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}) \\ & + \frac{1}{2}u_{s*}^2\left[\left(\nabla_{\mathrm{s}}^2\tau\right)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - 2\kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\right] + \frac{1}{2}u_{t*}u_{s*}\left[\left(\nabla_{\mathrm{s}}^2\tau\right)\boldsymbol{t}_*\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\right. \\ & \left.\left. + \left(\nabla_{\mathrm{s}}^2\tau\right)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t}_* - 2\kappa_g^*(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*)\right] - u'_{t*}u_{s*}\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\right\}\mathrm{d}s.\end{aligned} \tag{3.6}$$

In this formula, which is derived from (2.43) in appendix E, the field $\boldsymbol{u}$ has been decomposed along $\mathcal{C}$ as

$$\boldsymbol{u} = u_{t*}\boldsymbol{t}_* + u_{s*}\boldsymbol{\nu}_{\mathcal{S}_*}. \tag{3.7}$$

Our final concern in this section is to simplify the contributions to the second variation of $\mathcal{F}$ that arise along the contact line $\mathcal{C}$. Collecting the terms in (3.4), (3.5) and (3.6) that contain $\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}$, we see that

$$\int_{\mathcal{C}}\{\gamma(1+\cos\vartheta_c) - w - \tau\kappa_{g*} + (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})\}\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\mathrm{d}s = 0$$

by $(2.44)_2$. All remaining line integrals in equations (3.4), (3.5) and (3.6) constitute a functional in the single field $\boldsymbol{u}$ that we denote by $\frac{1}{2}\mathcal{F}_{\mathcal{C}}^{(2)}$. As shown in appendix F, this functional can

eventually be written as

$$\begin{aligned}\frac{1}{2}\mathcal{F}_{\mathcal{C}}^{(2)} = \int_{\mathcal{C}} \Bigg\{ &\left[\frac{\tau}{2}\kappa_g^{*2} - \frac{1}{2}\kappa_g^*(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}) + \frac{\gamma}{2}\sin\vartheta_c\kappa_n^* - \frac{\gamma}{2}\kappa_g + \frac{1}{2}(w-\gamma)\kappa_g^*\right]u_t^2 \\ &+\left[\frac{1}{2}\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \frac{1}{2}\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t} - \kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t} + \gamma\sin\vartheta_c(\tau_g^* - \tau_g)\right. \\ &\left.-\frac{1}{2}w' - \frac{\gamma}{2}(\cos\vartheta_c)'\right]u_t u_{s*} + \left[\tau\kappa_g^* - \frac{\gamma}{2}\cos\vartheta_c + \frac{1}{2}(w-\gamma)\right]u_t u_{s*}' \\ &+\left[\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \frac{\gamma}{2}\cos\vartheta_c - \frac{1}{2}(w-\gamma)\right]u_{s*}u_t' \\ &+\left[\frac{\tau}{2}\tau_g^{*2} - \frac{\tau}{2}\kappa_n^*\kappa_{n\perp}^* + \frac{1}{2}\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\right. \\ &+\frac{\gamma}{2}\sin\vartheta_c\kappa_{n\perp}^* - \frac{1}{2}(\nabla_{\mathrm{s}}w\cdot\boldsymbol{\nu}_{\mathcal{S}_*}) + \gamma\sin\vartheta_c\cos\vartheta_c\left(\frac{H}{2}+\kappa_{n\perp}\right) \\ &\left.-\frac{\gamma}{2}\kappa_g\cos^2\vartheta_c + \frac{1}{2}(w-\gamma)\kappa_g^*\right]u_{s*}^2 + \frac{\tau}{2}u_{s*}'^2\Bigg\}\,\mathrm{d}s. \end{aligned} \tag{3.8}$$

From (2.25), $(2.2)_2$, $(2.26)_1$ and the identity

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{t}_*}{\mathrm{d}s_*}$$

we see that

$$\kappa_g = \kappa_g^*\cos\vartheta_c + \kappa_n^*\sin\vartheta_c \tag{3.9}$$

so that in the integrand on the right-hand side of (3.8), the coefficient of u_t^2 becomes

$$\tfrac{1}{2}\kappa_g^*[\tau\kappa_g^* - \nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \gamma\cos\vartheta_c + w - \gamma]$$

which vanishes by $(2.44)_2$. Similarly, the integral along $\mathcal{C}$ of the three mixed terms vanishes, as we now proceed to show.

It is proved in appendix G that

$$\tau_g^* - \tau_g = -\frac{\mathrm{d}\vartheta_c}{\mathrm{d}s} = -\vartheta_c'. \tag{3.10}$$

Furthermore, by definition,

$$\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t} = (\nabla_{\mathrm{s}}\tau)' \tag{3.11}$$

and it follows from $(2.26)_2$ that

$$\frac{\mathrm{d}\boldsymbol{\nu}_{\mathcal{S}_*}}{\mathrm{d}s^*} = -\boldsymbol{\nu}_{\mathcal{S}_*}' = -\kappa_g^*\boldsymbol{t}_* - \tau_g^*\boldsymbol{\nu}_* = \kappa_g^*\boldsymbol{t} - \tau_g^*\boldsymbol{\nu}_*. \tag{3.12}$$

By (3.11) and (3.12), we easily see that

$$\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} = (\nabla_{\mathrm{s}}\tau)'\cdot\boldsymbol{\nu}_{\mathcal{S}_*} = (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})' + (\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t})\kappa_g^*.$$

Moreover, since

$$\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} = \big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t}$$

the coefficient of $u_t u_{s*}$ in the integrand on the right-hand side of (3.8) can be written as

$$\left[\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \frac{\gamma}{2}\cos\vartheta_c - \frac{1}{2}(w-\gamma)\right]'$$

while by $(2.44)_2$ the coefficient of $u_t u'_{s*}$ is

$$\nabla_{\rm s}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\frac{\gamma}{2}\cos\vartheta_c-\frac{1}{2}(w-\gamma).$$

Hence, the terms in $u_t u_{s*}$, $u_t u'_{s*}$, and $u'_t u_{s*}$ can be combined together as

$$\left\{\left[\nabla_{\rm s}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\frac{\gamma}{2}\cos\vartheta_c+\frac{1}{2}(w-\gamma)\right]u_t u_{s*}\right\}'$$

whose integral along $\mathcal{C}$ is nought.

Finally, further use of equation $(2.44)_2$ yields

$$\frac{w-\gamma}{2}\kappa_g^*=-\frac{\tau}{2}\kappa_g^{*2}+\frac{1}{2}\kappa_g^*\nabla_{\rm s}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\frac{\gamma}{2}\kappa_g^*\cos\vartheta_c.$$

Moreover, it follows from both $(F.5)_3$ and $(F.6)_3$ that

$$H=\operatorname{div}_{\rm s}\boldsymbol{\nu}=-(\kappa_n+\kappa_{n\perp})\qquad H^*=\operatorname{div}_{\rm s}\boldsymbol{\nu}_*=-(\kappa_n^*+\kappa_{n\perp}^*)$$
$$K^*=I_2(\nabla_{\rm s}\boldsymbol{\nu}_*)=\kappa_n^*\kappa_{n\perp}^*-\tau_g^{*2}.$$

Using these equations, (3.9), and its analogues

$$\kappa_g^*=\kappa_n\sin\vartheta_c+\kappa_g\cos\vartheta_c \tag{3.13}$$

$$\kappa_n^*=\kappa_g\sin\vartheta_c-\kappa_n\cos\vartheta_c \tag{3.14}$$

we then give (3.8) the concise form

$$\mathcal{F}_{\mathcal{C}}^{(2)}=\int_{\mathcal{C}}\left\{\tau(u'_{s*})^2-\gamma\beta u_{s*}^2\right\}\mathrm{d}s$$

where

$$\gamma\beta:=\tau\left(K^*+\kappa_g^{*2}\right)+(\nabla_{\rm s}w\cdot\boldsymbol{\nu}_{\mathcal{S}_*})-\left(\nabla_{\rm s}^2\tau\right)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\kappa_g^*\nabla_{\rm s}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}$$
$$+\gamma[H^*\sin\vartheta_c+H\cos\vartheta_c\sin\vartheta_c+\kappa_g\sin^2\vartheta_c]. \tag{3.15}$$

Thus, the complete expression for the second variation of $\mathcal{F}$ is reduced to

$$\delta^2\mathcal{F}=\gamma\int_{\mathcal{S}}\left\{|\nabla_{\rm s}u_\nu|^2+(2K-H^2+\partial_\nu f)u_\nu^2\right\}\mathrm{d}a+\int_{\mathcal{C}}\left\{\tau(u'_{s*})^2-\gamma\beta u_{s*}^2\right\}\mathrm{d}s \tag{3.16}$$

which, by (3.2) is equivalent to the desired result (3.1).

4. Stability criterion

In the preceding section we have recorded the main steps of the laborious computation required to write effectively the second variation of the general energy functional $\mathcal{F}$ in (2.9)–(2.13). Here we derive from equation (3.16) a criterion for the local stability of the equilibrium configurations of a drop described by $\mathcal{F}$. Precisely, 'local stability' for the drop means 'strong positiveness' for $\delta^2\mathcal{F}$, in the same sense recalled by [7], but here relative to the L^2-norm over $\mathcal{S}$. This requirement is equivalent to subjecting $\delta^2\mathcal{F}$ to the non-homogeneous constraint

$$\int_{\mathcal{S}}u_\nu^2\,\mathrm{d}a=1 \tag{4.1}$$

in addition to $(2.29)_1$, and to requiring $\delta^2\mathcal{F}$ to attain a positive minimum. As is standard in such a case (see section VI.1 of [23]), we define the functional

$$F[u_\nu]:=\frac{1}{2}\int_{\mathcal{S}}\left\{|\nabla_{\rm s}u_\nu|^2+\alpha u_\nu^2\right\}\mathrm{d}a+\lambda\int_{\mathcal{S}}u_\nu\,\mathrm{d}a-\frac{1}{2}\mu\int_{\mathcal{S}}u_\nu^2\,\mathrm{d}a+\frac{1}{2}\int_{\mathcal{C}}\left\{\xi(u'_{s*})^2-\beta u_{s*}^2\right\}\mathrm{d}s \tag{4.2}$$

where

$$\alpha := 2K - H^2 + \partial_\nu f \tag{4.3}$$

$$\xi := \frac{\tau}{\gamma} \tag{4.4}$$

and λ and μ are Lagrange multipliers associated with $(2.29)_1$ and (4.1), respectively.

We now write the equilibrium equations for F, which will lead us to identify μ with the minimum value attained by F on the constraints $(2.29)_1$ and (4.1). Recalling (3.2), we set

$$u_{s*} = \chi u_\nu \qquad \text{on } \mathcal{C} \qquad \text{with} \quad \chi := \frac{1}{\sin\vartheta_c}. \tag{4.5}$$

Perturbing u_ν as below

$$u_\nu^\varepsilon = u_\nu + \varepsilon\eta \tag{4.6}$$

with η a smooth scalar function on $\mathcal{S}$, induces a similar perturbation in u_{s*} on $\mathcal{C}$:

$$u_{s*}^\varepsilon = u_{s*} + \varepsilon\chi\eta. \tag{4.7}$$

It readily follows from (4.2) that

$$\delta F[\eta] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F\big[u_\nu^\varepsilon\big]\Big|_{\varepsilon=0} = \int_{\mathcal{S}} \{\nabla_{\mathrm{s}} u_\nu \cdot \nabla_{\mathrm{s}}\eta + [(\alpha-\mu)u_\nu + \lambda]\eta\}\,\mathrm{d}a + \int_{\mathcal{C}} \{\xi u'_{s*}(\chi\eta)' - \beta u_{s*}\chi\eta\}\,\mathrm{d}s. \tag{4.8}$$

By applying the surface-divergence theorem to the integral over $\mathcal{S}$ in (4.8) and performing an integration by parts both there and in the integral along $\mathcal{C}$, we easily see that δF vanishes identically whenever u_ν obeys the equations

$$\Delta_{\mathrm{s}} u_\nu - \alpha u_\nu - \lambda + \mu u_\nu = 0 \qquad \text{on } \mathcal{S} \tag{4.9}$$

$$\nabla_{\mathrm{s}} u_\nu \cdot \boldsymbol{\nu}_{\mathcal{S}} - \chi(\xi(\chi u_\nu)')' - \beta\chi^2 u_\nu = 0 \qquad \text{along } \mathcal{C}. \tag{4.10}$$

Now suppose that u_ν is a solution to these equations with λ and μ chosen such that $(2.29)_1$ and (4.1) are satisfied. By multiplying both sides of equation (4.9) by u_ν and those of equation (4.10) by u_ν and then integrating over $\mathcal{S}$ and along $\mathcal{C}$, respectively, also using both $(2.29)_1$ and (4.1), we arrive at

$$F[u_\nu] = \mu.$$

This shows that the least value of μ for which there is a solution to equations (4.9) and (4.10) is also the minimum of F on the constraints $(2.29)_1$ and (4.1). We thus conclude that an equilibrium configuration of a drop with energy represented as in (2.9)–(2.13) is locally stable whenever the least eigenvalue μ in (4.9) is positive. At first glance, only the free surface $\mathcal{S}$ and the contact line $\mathcal{C}$ contribute directly to determine the stable configurations of a wetting drop. A deeper insight, however, reveals that the wetted surface $\mathcal{S}_*$ of the substrate plays an indirect role in the drop stability through both the total and Gaussian curvatures at the points traversed by the contact line $\mathcal{C}$ and through the geodesic curvature of $\mathcal{C}$ relative to $\mathcal{S}_*$ (cf equation (3.15)).

Here only the stability of the functional $\mathcal{F}$ has been systematically addressed; the issue as to whether our stability criterion is related to the minimality of $\mathcal{F}$ was left untouched. We conjecture that if the least eigenvalue that solves the problem in $L^2(\mathcal{S})$ stated in this section is positive, then the second variation of $\mathcal{F}$ is also strongly positive in $H^1(\mathcal{S})$, and hence $\mathcal{F}$ attains a strict local minimum at $\mathcal{S}$. We still do not possess a formal proof of this conjecture. Were it true, our stability criterion for $\mathcal{F}$ would also ensure minimality.

5. Conclusions and comments

We computed the second variation of the energy functional for an incompressible liquid drop wetting a possibly structured substrate. We endeavoured to treat the substrate as arbitrarily curved, and to consider a sufficiently general energy functional so that diluted interface potentials could also be considered in our model. In principle, this makes our model applicable to length scales where the line tension τ of the drop, that is, Gibbs's excess energy residing along the contact line, plays a role in the drop's equilibrium configurations. The characteristic length-scale ξ for the influence of line tension is apparently defined by (4.4). Since a theoretical estimate for τ is 10^{-11} N (see, for example, [24] and p 240 of [25]) and γ is usually of the order of 10^{-2} N m^{-1}, ξ is expected to be in the nanometre range, where the validity of the continuum model adopted in this paper could easily be questioned. Nonetheless, the line tension that defines ξ is the most controversial quantity in the whole of wetting science. For a review of the many problems connected with this quantity, starting from its very definition, we refer the reader to [26]. As a reference figure, we record here the estimate made in [27] which shows that for τ of the order of 10^{-9} N, the line tension should have a noticeable effect on the wetting morphology of a drop when its size is 300 nm or smaller. Disparate measurements of τ have appeared in the literature, some higher even by several orders of magnitude than the theoretical estimates just recalled (see, for example, the values recorded in [28, 29]). Moreover, τ can be either positive or negative.

Recently, very accurate measurements have shown that the line tension actually ranges between 10^{-10} and 10^{-11} N (see, for example, [30–32]). In particular, in the experiments performed by Wand *et al* [30], τ was found to change its sign with the temperature.

This neat result contradicts the theoretical argument of Li and Steigmann in [28] (see also [29, 33]) claiming that thermodynamic stability requires τ to be positive. A similar argument was also given by Clarke [34] within a statistical theory of capillarity. In essence, these arguments assert that the drop's energy functional $\mathcal{F}$ in (2.9) is unbounded from below if $\tau < 0$, as a wigglier and wigglier contact line $\mathcal{C}$ would decrease the line integral $\mathcal{F}_\ell$ further and further, almost without affecting the integrals $\mathcal{F}_b$, $\mathcal{F}_a$ and $\mathcal{F}_s$. However, while this is unmistakably true, one should heed that in the limit where the curvature of $\mathcal{C}$ diverges the model employed here also becomes inappropriate, as all curvature corrections to both surface and line energies are ignored in $\mathcal{F}$ (see [35] for a formal treatment of these terms): this is likely to undermine any formal instability argument resorting to a contact line that locally becomes infinitely curved.

In our opinion, the stability criterion arrived at in this paper has the potential to clarify this matter. Be τ positive or negative, the real issue is to find the shortest length d over which the drop's shape could be perturbed while leaving $\delta^2\mathcal{F}$ positive: the given equilibrium configuration of the drop is actually unstable against modes that deform it over length scales shorter than d. A partial conclusion has already been attained with this approach for straight liquid ridges with radius R and contact angle $\vartheta_c = \frac{\pi}{2}$ on a flat substrate: if $\tau < 0$ then $d \approx \sqrt{|\xi|R}$ (see [36]). This length could be much larger than $|\xi|$, which thus, at least in this special case, is by no means a length scale that a drop described by the functional $\mathcal{F}$ in section 2 can actually experience when $\tau < 0$.

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Appendix A. Proof of lemma 2.2

Standard rules of tensor analysis (see, e.g., [37]) yield

$$\operatorname{div}_s[(\nabla_s \boldsymbol{g})\boldsymbol{g} - (\operatorname{div}_s \boldsymbol{g})\boldsymbol{g}] = \operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}} \cdot \boldsymbol{g} + \operatorname{tr}(\nabla_s \boldsymbol{g})^2 - (\operatorname{div}_s \boldsymbol{g})^2 - \nabla_s(\operatorname{div}_s \boldsymbol{g}) \cdot \boldsymbol{g}$$

and so, by definition (2.8),

$$I_2(\nabla_s \boldsymbol{g}) = \tfrac{1}{2}[\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}} - \nabla_s(\operatorname{div}_s \boldsymbol{g})] \cdot \boldsymbol{g} - \tfrac{1}{2} \operatorname{div}_s[(\nabla_s \boldsymbol{g})\boldsymbol{g} - (\operatorname{div}_s \boldsymbol{g})\boldsymbol{g}]. \tag{A.1}$$

To give the quantity $[\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}} - \nabla_s(\operatorname{div}_s \boldsymbol{g})] \cdot \boldsymbol{g}$ a simpler expression, we temporarily resort to Cartesian components. The mth component of the vector $\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}}$ in the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is

$$[\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}}]_m := \operatorname{div}_s[(\nabla_s \boldsymbol{g})\boldsymbol{e}_m].$$

The components of the vector $\boldsymbol{d} := (\nabla_s \boldsymbol{g})\boldsymbol{e}_m$ are

$$d_i = g_{i,k} P_{km}$$

where the sum over repeated indices is understood,

$$P_{jk} := \delta_{jk} - \nu_j \nu_k \tag{A.2}$$

are the components of the projector onto the plane orthogonal to $\boldsymbol{\nu}$, and $g_{i,k} := \frac{\partial g_i}{\partial x_k}$ are the Cartesian components of $\nabla \boldsymbol{g}$, where the gradient is evaluated on an extension of $\boldsymbol{g}$ away from the surface $\mathcal{S}$. Hence,

$$(\operatorname{div}_s \boldsymbol{d})_m = (g_{i,k} P_{km})_{,j} P_{ji} = g_{i,kj} P_{km} P_{ji} - g_{i,k}(\nu_{k,j} \nu_m + \nu_k \nu_{m,j}) P_{ji}.$$

Furthermore,

$$[\nabla_s(\operatorname{div}_s \boldsymbol{g})]_m = P_{mk}[\operatorname{div}_s \boldsymbol{g}]_{,k} = P_{mk} g_{i,jk} P_{ji} - P_{mk} g_{i,j}[\nu_{i,k} \nu_j + \nu_{j,k} \nu_i]$$

so that by (A.2) and the symmetry of $g_{i,jk}$ in the indices j and k, we obtain that

$$\begin{aligned}[(\operatorname{div}_s \boldsymbol{d}) - \nabla_s(\operatorname{div}_s \boldsymbol{g})]_m &= -g_{i,k} \nu_{k,i} \nu_m - g_{i,k} \nu_k \nu_{m,i} \\ &\quad + \nu_i \nu_j \nu_k g_{i,k} \nu_{m,j} + g_{i,j} \nu_j \nu_{i,m} + g_{i,j} \nu_i \nu_{j,m} - g_{i,j} \nu_m \nu_j \nu_k \nu_{i,k}.\end{aligned} \tag{A.3}$$

Thus, rewriting (A.3) in intrinsic notation, we arrive at

$$\begin{aligned}\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}} - \nabla_s(\operatorname{div}_s \boldsymbol{g}) &= -[(\nabla \boldsymbol{\nu})^{\mathsf{T}} \cdot \nabla \boldsymbol{g}]\boldsymbol{\nu} + ((\nabla \boldsymbol{g})\boldsymbol{\nu} \cdot \boldsymbol{\nu})(\nabla \boldsymbol{\nu})\boldsymbol{\nu} \\ &\quad + [(\nabla \boldsymbol{\nu})^{\mathsf{T}} - (\nabla \boldsymbol{\nu})](\nabla \boldsymbol{g})\boldsymbol{\nu} + (\nabla \boldsymbol{\nu})^{\mathsf{T}}(\nabla \boldsymbol{g})^{\mathsf{T}}\boldsymbol{\nu} + [(\nabla \boldsymbol{\nu})\boldsymbol{\nu} \cdot (\nabla \boldsymbol{g})\boldsymbol{\nu}]\boldsymbol{\nu}\end{aligned} \tag{A.4}$$

where the full gradient of $\boldsymbol{\nu}$ is computed on an extension of this field away from the surface $\mathcal{S}$. Since $\nabla \boldsymbol{\nu}$ and the intrinsic surface gradient $\nabla_s \boldsymbol{\nu}$ are related through

$$\nabla \boldsymbol{\nu} = \nabla_s \boldsymbol{\nu} + (\nabla \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu}$$

and $\nabla_s \boldsymbol{\nu}$ is symmetric, we have that

$$\begin{aligned}[(\nabla \boldsymbol{\nu})^{\mathsf{T}} - (\nabla \boldsymbol{\nu})](\nabla \boldsymbol{g})\boldsymbol{\nu} \cdot \boldsymbol{g} &= [\boldsymbol{\nu} \otimes \boldsymbol{\nu}(\nabla \boldsymbol{\nu})^{\mathsf{T}} - (\nabla \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu}](\nabla \boldsymbol{g})\boldsymbol{\nu} \cdot \boldsymbol{g} \\ &= -[(\nabla \boldsymbol{\nu})\boldsymbol{\nu} \cdot \boldsymbol{g}](\nabla \boldsymbol{g})\boldsymbol{\nu} \cdot \boldsymbol{\nu}\end{aligned}$$

where we also used the hypothesis that

$$\boldsymbol{g} \cdot \boldsymbol{\nu} = 0. \tag{A.5}$$

Hence, by the identity

$$\nabla_s \boldsymbol{g} = \nabla \boldsymbol{g}(\mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})$$

we obtain that

$$[\operatorname{div}_s(\nabla_s \boldsymbol{g})^{\mathsf{T}} - \nabla_s(\operatorname{div}_s \boldsymbol{g})] \cdot \boldsymbol{g} = (\nabla \boldsymbol{\nu})^{\mathsf{T}}(\nabla \boldsymbol{g})^{\mathsf{T}}\boldsymbol{\nu} \cdot \boldsymbol{g} = -|(\nabla_s \boldsymbol{\nu})\boldsymbol{g}|^2$$

where we have resorted to both (A.5) and its consequence

$$(\nabla_s \boldsymbol{g})^{\mathsf{T}} \boldsymbol{\nu} = -(\nabla_s \boldsymbol{\nu})\boldsymbol{g}. \tag{A.6}$$

Thus, (A.1) becomes

$$I_2(\nabla_s \boldsymbol{g}) = -\tfrac{1}{2}\{|(\nabla_s \boldsymbol{\nu})\boldsymbol{g}|^2 + \mathrm{div}_s[(\nabla_s \boldsymbol{g})\boldsymbol{g} - (\mathrm{div}_s\, \boldsymbol{g})\boldsymbol{g}]\}. \tag{A.7}$$

By (A.5), application of the surface-divergence theorem leads us to

$$\int_{\mathcal{S}} I_2(\nabla_s \boldsymbol{g})\, \mathrm{d}a = -\frac{1}{2}\int_{\mathcal{S}} |(\nabla_s \boldsymbol{\nu})\boldsymbol{g}|^2 \mathrm{d}a - \frac{1}{2}\int_{\mathcal{S}} H(\nabla_s \boldsymbol{g})\boldsymbol{g}\cdot\boldsymbol{\nu}\, \mathrm{d}a$$
$$-\frac{1}{2}\int_{\mathcal{C}} [(\nabla_s \boldsymbol{g})\boldsymbol{g} - (\mathrm{div}_s\, \boldsymbol{g})\boldsymbol{g}]\cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s.$$

Furthermore, it follows from lemma 2.1 that

$$H\boldsymbol{g}\cdot(\nabla_s \boldsymbol{\nu})\boldsymbol{g} = K|\boldsymbol{g}|^2 + |(\nabla_s \boldsymbol{\nu})\boldsymbol{g}|^2 \tag{A.8}$$

whence further application of (A.6) yields (2.7).

Appendix B. Second-order variation

In this appendix we prove the equations in (2.27), which played a central role in this paper.

Before giving a general proof, we illustrate by example the subtleties implied in finding the minima of a function on a manifold. Let $\mathcal{C}$ be a planar curve, parametrized in its arc-length s as

$$p(s) - o = x(s)\boldsymbol{e}_x + y(s)\boldsymbol{e}_y$$

and let ϑ be the angle that the unit tangent vector $\boldsymbol{t}$ to $\mathcal{C}$ makes with $\boldsymbol{e}_x$. Then

$$\boldsymbol{t} = \cos\vartheta\, \boldsymbol{e}_x + \sin\vartheta\, \boldsymbol{e}_y$$

and so

$$x' = \cos\vartheta \qquad y' = \sin\vartheta$$

where a prime denotes differentiation with respect to s. Moreover, the principal unit normal $\boldsymbol{n}$ to $\mathcal{C}$ is

$$\boldsymbol{n} = -\sin\vartheta\, \boldsymbol{e}_x + \cos\vartheta\, \boldsymbol{e}_y$$

so that

$$\boldsymbol{t}' = \sigma \boldsymbol{n} \tag{B.1}$$

where $\sigma = \vartheta'$ is the curvature of $\mathcal{C}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let

$$s \mapsto g(s) := f(x(s), y(s))$$

be the restriction of f to $\mathcal{C}$. Clearly, a way to find the stationary points of f on $\mathcal{C}$ is to find the stationary points of g. We thus compute

$$g' = f_x x' + f_y y' = \nabla f \cdot \boldsymbol{t}.$$

The points where $g' = 0$ are those where ∇f is orthogonal to $\mathcal{C}$. At any of these points f attains a local minimum whenever $g'' > 0$. By (B.1), we easily see that this condition reads

$$g'' = \boldsymbol{t}\cdot(\nabla^2 f)\boldsymbol{t} + \sigma \nabla f \cdot \boldsymbol{n} > 0. \tag{B.2}$$

This same conclusion can also be reached otherwise: by exploring f near a putative stationary point p_0 on $\mathcal{C}$ by means of a local perturbation of p_0 designed so as to describe a

small arc on $\mathcal{C}$. With this aim we note that, at least locally, we can represent the curve $\mathcal{C}$ as the level set of a function h, so that $p \in \mathcal{C}$ if and only if $h(p) = 0$. We perturb p_0 as follows:

$$p_\varepsilon = p_0 + \varepsilon \boldsymbol{u} + \varepsilon^2 \boldsymbol{v}$$

and we require that p_ε belongs to $\mathcal{C}$, up to *second-order* terms in ε. To find the restrictions imposed on $\boldsymbol{u}$ and $\boldsymbol{v}$ by this requirement, we first expand h near p_0 as follows:

$$h(p_\varepsilon) = h(p_0) + \varepsilon(\nabla h \cdot \boldsymbol{u}) + \tfrac{1}{2}\varepsilon^2[\boldsymbol{u} \cdot (\nabla^2 h)\boldsymbol{u} + 2\nabla h \cdot \boldsymbol{v}] + O(\varepsilon^3) \tag{B.3}$$

and then we set $h(p_\varepsilon) = O(\varepsilon^3)$. Since p_0 belongs to $\mathcal{C}$ and $\nabla h = |\nabla h|\boldsymbol{n}$, requiring the right-hand side of (B.3) to vanish up to first order in ε amounts to requiring that

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{B.4}$$

or, equivalently, that $\boldsymbol{u} = u\boldsymbol{t}$. Moreover, using the Frénet–Serret equation for planar curves

$$\boldsymbol{n}' = -\sigma \boldsymbol{t}$$

we arrive at

$$(\nabla^2 h)\boldsymbol{t} = |\nabla h|'\boldsymbol{n} - |\nabla h|\sigma \boldsymbol{t}$$

whence it follows that the term in ε^2 also vanishes in (B.3), provided that

$$v_v := \boldsymbol{v} \cdot \boldsymbol{n} = \frac{\sigma}{2} u^2. \tag{B.5}$$

To characterize the stationary points of f on $\mathcal{C}$ we expand f up to second order, obtaining

$$f(p_\varepsilon) = f(p_0) + \varepsilon(\nabla f \cdot \boldsymbol{u}) + \tfrac{1}{2}\varepsilon^2[\boldsymbol{u} \cdot (\nabla^2 f)\boldsymbol{u} + 2\nabla f \cdot \boldsymbol{v}] + O(\varepsilon^3). \tag{B.6}$$

The function f is stationary at p_0, provided that

$$\nabla f \cdot \boldsymbol{u} = 0$$

or, equivalently, by (B.4), provided that

$$\nabla f \cdot \boldsymbol{t} = 0.$$

Finally, for f to attain a minimum at p_0 the quadratic term in (B.6) must be positive for all admissible $\boldsymbol{u}$ and $\boldsymbol{v}$. Since, by (B.4) and (B.5),

$$\boldsymbol{u} \cdot (\nabla^2 f)\boldsymbol{u} + 2\nabla f \cdot \boldsymbol{v} = u^2 \boldsymbol{t} \cdot (\nabla^2 f)\boldsymbol{t} + 2v_n \nabla f \cdot \boldsymbol{n} = u^2[\boldsymbol{t} \cdot (\nabla^2 f)\boldsymbol{t} + \sigma \nabla f \cdot \boldsymbol{n}]$$

we easily retrace the same inequality in (B.2).

The proof of both equations in (2.27) is indeed a slight extension of the example just discussed. Let a smooth orientable surface $\mathcal{S}_*$ be locally represented as a level set of the scalar function h, so that $h(p) = 0$ and the unit normal $\boldsymbol{\nu}_*$ to $\mathcal{S}_*$ is represented as

$$\boldsymbol{\nu}_* = \frac{\nabla h}{|\nabla h|}. \tag{B.7}$$

We map a point p on $\mathcal{S}_*$ into

$$p_\varepsilon = p + \varepsilon \boldsymbol{u} + \varepsilon^2 \boldsymbol{v}$$

and we seek the conditions that both fields $\boldsymbol{u}$ and $\boldsymbol{v}$ must obey to guarantee that p_ε belongs to $\mathcal{S}_*$, up to second-order terms in ε^2, that is, to ensure that $h(p_\varepsilon) = O(\varepsilon^3)$. By repeating *verbatim* the reasoning leading to (B.3), we readily arrive at in the following conditions:

$$\nabla h \cdot \boldsymbol{u} = 0 \tag{B.8}$$

and

$$\boldsymbol{u} \cdot (\nabla^2 h)\boldsymbol{u} + 2\nabla h \cdot \boldsymbol{v} = 0. \tag{B.9}$$

By (B.7), (B.8) is equivalent to $(2.27)_1$. Moreover, we note that

$$\nabla(|\nabla h|) = \nabla(\nabla h \cdot \nabla h)^{1/2} = (\nabla^2 h)\boldsymbol{\nu}_*$$

because $\nabla^2 h$ is a symmetric tensor. For the same reason we can write

$$\nabla\boldsymbol{\nu}_* = \frac{1}{|\nabla h|}\nabla^2 h - \nabla h \otimes \frac{1}{|\nabla h|^2}\nabla(|\nabla h|) = \frac{1}{|\nabla h|}\mathbf{P}\nabla^2 h \tag{B.10}$$

where $\mathbf{P} := \mathbf{I} - \boldsymbol{\nu}_* \otimes \boldsymbol{\nu}_*$ is the projector onto the plane tangent to $\mathcal{S}_*$ at p_0. The full gradient of $\boldsymbol{\nu}_*$ in (B.10) is indeed the gradient of the extension of $\boldsymbol{\nu}_*$ in (B.7); it is related to the intrinsic surface gradient $\nabla_{\mathrm{s}}\boldsymbol{\nu}_*$ by the formula

$$\nabla_{\mathrm{s}}\boldsymbol{\nu}_* = (\nabla\boldsymbol{\nu}_*)\mathbf{P}. \tag{B.11}$$

Since $\boldsymbol{u}$ is tangential and $\mathbf{P}$ is symmetric we can write

$$\boldsymbol{u} \cdot (\nabla^2 h)\boldsymbol{u} = \mathbf{P}\boldsymbol{u} \cdot (\nabla^2 h)\mathbf{P}\boldsymbol{u} = \boldsymbol{u} \cdot \mathbf{P}(\nabla^2 h)\mathbf{P}\boldsymbol{u}$$

whence, by (B.10) and (B.11), it follows that

$$\frac{1}{|\nabla h|}\boldsymbol{u} \cdot (\nabla^2 h)\boldsymbol{u} = \frac{1}{|\nabla h|}\boldsymbol{u} \cdot (\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}. \tag{B.12}$$

Thus, also by (B.7), (B.9) becomes

$$\boldsymbol{u} \cdot (\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u} + 2\boldsymbol{v} \cdot \boldsymbol{\nu}_* = 0$$

which is precisely equation $(2.27)_2$.

Appendix C. Second variation of $\mathcal{F}_b + \gamma A$

From equations (2.21) and (2.36) we see that

$$\begin{aligned}\frac{1}{2}\delta^2\mathcal{F}_b &= \int_{\mathcal{S}} \left\{ f\left[v_\nu + \frac{1}{2}(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})u_\nu - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{a}\right] + \frac{1}{2}[(\nabla f) \cdot \boldsymbol{u}]u_\nu \right\} \mathrm{d}a \\ &= E_1 + E_2 + E_3 + E_4 \end{aligned} \tag{C.1}$$

where we have set

$$E_1 := \int_{\mathcal{S}} f v_\nu \,\mathrm{d}a \qquad E_2 := \frac{1}{2}\int_{\mathcal{S}} f[(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})u_\nu - \boldsymbol{u} \cdot \boldsymbol{a}]\,\mathrm{d}a \tag{C.2}$$

$$E_3 := \frac{1}{2}\int_{\mathcal{S}} u_\nu(\nabla_{\mathrm{s}} f) \cdot \boldsymbol{u}_\parallel \,\mathrm{d}a \qquad E_4 := \frac{1}{2}\int_{\mathcal{S}} (\partial_\nu f)u_\nu^2 \,\mathrm{d}a. \tag{C.3}$$

Then, in view of (2.37), we find it convenient to compute separately $\gamma\delta^2 A$; with the aid of $(2.27)_2$, (2.33) and (2.25), one readily arrives at

$$\frac{1}{2}\gamma\delta^2 A = \gamma\int_{\mathcal{S}} \left[H v_\nu + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u})\right] \mathrm{d}a + E_5 \tag{C.4}$$

where

$$E_5 := \gamma\int_{\mathcal{C}} \left\{ (1 + \cos\vartheta_c)\boldsymbol{v} \cdot \boldsymbol{\nu}_{\mathcal{S}_*} - \frac{1}{2}[(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u} - (\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u}] \cdot \boldsymbol{\nu}_{\mathcal{S}_*} - \frac{1}{2}\sin\vartheta_c \boldsymbol{u} \cdot (\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u} \right\} \mathrm{d}s \tag{C.5}$$

collects the contributions to $\frac{1}{2}\gamma\delta^2 A$ localized on $\mathcal{C}$. Combining together all surface integrals in both (C.1) and (C.4), we define

$$E_6 := \gamma \int_{\mathcal{S}} \left[H v_\nu + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) \right] \mathrm{d}a + E_1 + E_2$$
$$= \int_{\mathcal{S}} \left[(\gamma H + f) v_\nu + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) \right] \mathrm{d}a + E_2.$$

By using the equilibrium equation $(2.44)_1$ we also give E_6 in the following form:

$$E_6 = \int_{\mathcal{S}} \left[\lambda v_\nu + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) \right] \mathrm{d}a + E_2$$

which can be thus simplified by resorting to $(2.29)_2$:

$$E_6 = \int_{\mathcal{S}} \left[\frac{\lambda}{2}(\boldsymbol{u}\cdot\boldsymbol{a} - u_\nu \operatorname{div}_{\mathrm{s}} \boldsymbol{u}) + \frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) \right] \mathrm{d}a + E_2.$$

Finally, recalling $(\mathrm{C}.2)_2$ and $(2.44)_1$, we conclude that

$$E_6 = \gamma \int_{\mathcal{S}} \left\{ \frac{H}{2}(\boldsymbol{u}\cdot\boldsymbol{a} - u_\nu \operatorname{div}_{\mathrm{s}} \boldsymbol{u}) + \left[\frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) \right] \right\} \mathrm{d}a$$

and we note that now

$$\tfrac{1}{2}(\delta^2 \mathcal{F}_b + \gamma\delta^2 A) = E_3 + E_4 + E_5 + E_6. \tag{C.6}$$

To simplify E_6 further we use equation (3.3) and write

$$\nabla_{\mathrm{s}}\boldsymbol{u} = \nabla_{\mathrm{s}}\boldsymbol{u}_\parallel + \boldsymbol{\nu} \otimes \nabla_{\mathrm{s}} u_\nu + u_\nu \nabla_{\mathrm{s}}\boldsymbol{\nu} \tag{C.7}$$

so that, by (2.21), $\boldsymbol{a}$ can be decomposed as

$$\boldsymbol{a} = \boldsymbol{a}_\parallel + \nabla_{\mathrm{s}} u_\nu \tag{C.8}$$

where $\boldsymbol{a}_\parallel := (\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)^{\mathsf{T}}\boldsymbol{\nu}$. Since $\boldsymbol{u}_\parallel \cdot \boldsymbol{\nu} = 0$ and $\nabla_{\mathrm{s}}\boldsymbol{\nu}$ is a symmetric tensor, we also have that

$$\boldsymbol{a}_\parallel = -(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_\parallel. \tag{C.9}$$

Moreover, again by the symmetry of $\nabla_{\mathrm{s}}\boldsymbol{\nu}$,

$$(\nabla_{\mathrm{s}}\boldsymbol{u})^2 = (\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)^2 + u_\nu(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)(\nabla_{\mathrm{s}}\boldsymbol{\nu}) + \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)^{\mathsf{T}}(\nabla_{\mathrm{s}} u_\nu)$$
$$+ u_\nu \boldsymbol{\nu} \otimes (\nabla_{\mathrm{s}}\boldsymbol{\nu})\nabla_{\mathrm{s}} u_\nu + u_\nu(\nabla_{\mathrm{s}}\boldsymbol{\nu})(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel) + u_\nu^2(\nabla_{\mathrm{s}}\boldsymbol{\nu})^2$$

whence, by (2.4), it follows that

$$\operatorname{tr}(\nabla_{\mathrm{s}}\boldsymbol{u})^2 = \operatorname{tr}(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)^2 + 2u_\nu \nabla_{\mathrm{s}}\boldsymbol{\nu} \cdot \nabla_{\mathrm{s}}\boldsymbol{u}_\parallel + (H^2 - 2K)u_\nu^2.$$

Since, by (C.7),

$$\operatorname{div}_{\mathrm{s}} \boldsymbol{u} = \operatorname{div}_{\mathrm{s}} \boldsymbol{u}_\parallel + H u_\nu \tag{C.10}$$

we conclude that

$$I_2(\nabla_{\mathrm{s}}\boldsymbol{u}) = I_2(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel) + K u_\nu^2 + u_\nu(H \operatorname{div}_{\mathrm{s}} \boldsymbol{u}_\parallel - \nabla_{\mathrm{s}}\boldsymbol{\nu} \cdot \nabla_{\mathrm{s}}\boldsymbol{u}_\parallel).$$

Similarly, resorting to (C.8) leads us to

$$E_6 = \gamma \int_{\mathcal{S}} \left[\frac{H}{2}(\boldsymbol{u}\cdot\boldsymbol{a} - u_\nu \operatorname{div}_{\mathrm{s}} \boldsymbol{u}) \right] \mathrm{d}a + \int_{\mathcal{S}} \left[\frac{1}{2}|\boldsymbol{a}_\parallel|^2 + \frac{1}{2}|\nabla_{\mathrm{s}} u_\nu|^2 + \boldsymbol{a}_\parallel \cdot \nabla_{\mathrm{s}} u_\nu \right.$$
$$\left. + I_2(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel) + K u_\nu^2 + u_\nu(H \operatorname{div}_{\mathrm{s}} \boldsymbol{u}_\parallel - \nabla_{\mathrm{s}}\boldsymbol{\nu} \cdot \nabla_{\mathrm{s}}\boldsymbol{u}_\parallel) \right] \mathrm{d}a. \tag{C.11}$$

Perusal of (2.21), (3.3), (C.8), (C.9), (C.10) and lemma 2.2 transforms (C.11) into

$$E_6 = \gamma \int_{\mathcal{S}} \left\{ \frac{1}{2}(\nabla_{\mathrm{s}}\boldsymbol{\nu})^2 \boldsymbol{u}_{\parallel} \cdot \boldsymbol{u}_{\parallel} + \frac{1}{2}|\nabla_{\mathrm{s}} u_\nu|^2 + \frac{K}{2}\boldsymbol{u}_{\parallel} \cdot \boldsymbol{u}_{\parallel} + \frac{1}{2}(2K - H^2)u_\nu^2 \right.$$
$$- \frac{H}{2}\boldsymbol{u}_{\parallel} \cdot (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel} + \frac{H}{2}(u_\nu \mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}_{\parallel} + \boldsymbol{u}_{\parallel} \cdot \nabla_{\mathrm{s}} u_\nu) - u_\nu \nabla_{\mathrm{s}}\boldsymbol{\nu} \cdot \nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel}$$
$$\left. - (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel} \cdot \nabla_{\mathrm{s}} u_\nu \right\} \mathrm{d}a - \frac{\gamma}{2}\int_{\mathcal{C}} [(\nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel} - (\mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel}] \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s.$$

By the following identities:

$$\frac{H}{2}[u_\nu\, \mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}_{\parallel} + \boldsymbol{u}_{\parallel} \cdot \nabla_{\mathrm{s}} u_\nu] = \frac{H}{2}\, \mathrm{div}_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel})$$
$$u_\nu \nabla_{\mathrm{s}}\boldsymbol{\nu} \cdot \nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel} + (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel} \cdot \nabla_{\mathrm{s}} u_\nu = (\nabla_{\mathrm{s}}\boldsymbol{\nu}) \cdot \nabla_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel})$$

we can also give E_6 the expression

$$E_6 = \gamma \int_{\mathcal{S}} \frac{1}{2}[(\nabla_{\mathrm{s}}\boldsymbol{\nu})^2 - H\nabla_{\mathrm{s}}\boldsymbol{\nu} + K\mathbf{I}]\boldsymbol{u}_{\parallel} \cdot \boldsymbol{u}_{\parallel}\, \mathrm{d}a + \frac{\gamma}{2}\int_{\mathcal{S}} \left[|\nabla_{\mathrm{s}} u_\nu|^2 + (2K - H^2)u_\nu^2\right] \mathrm{d}a$$
$$+ \gamma \int_{\mathcal{S}} \frac{H}{2}\mathrm{div}_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel})\, \mathrm{d}a - \gamma \int_{\mathcal{S}} (\nabla_{\mathrm{s}}\boldsymbol{\nu}) \cdot \nabla_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel})\, \mathrm{d}a$$
$$- \frac{\gamma}{2}\int_{\mathcal{C}} [(\nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel} - (\mathrm{div}_{\mathrm{s}}\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel}] \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s. \tag{C.12}$$

Since $\boldsymbol{u}_{\parallel}$ is a tangent vector field, the first integral in (C.12) vanishes by lemma 2.1. Furthermore, the surface-divergence theorem yields

$$\int_{\mathcal{S}} \frac{H}{2}\, \mathrm{div}_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel}) = -\frac{1}{2}\int_{\mathcal{S}} u_\nu \nabla_{\mathrm{s}} H \cdot \boldsymbol{u}_{\parallel}\, \mathrm{d}a + \frac{1}{2}\int_{\mathcal{C}} H u_\nu \boldsymbol{u}_{\parallel} \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s.$$

Likewise, since $\nabla_{\mathrm{s}}\boldsymbol{\nu}$ is a symmetric tensor,

$$(\nabla_{\mathrm{s}}\boldsymbol{\nu}) \cdot \nabla_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel}) = \mathrm{div}_{\mathrm{s}}[(\nabla_{\mathrm{s}}\boldsymbol{\nu})u_\nu \boldsymbol{u}_{\parallel}] - u_\nu \Delta_{\mathrm{s}}\boldsymbol{\nu} \cdot \boldsymbol{u}_{\parallel}$$

whence, by the surface-divergence theorem and (2.5), we arrive at

$$\int_{\mathcal{S}} (\nabla_{\mathrm{s}}\boldsymbol{\nu}) \cdot \nabla_{\mathrm{s}}(u_\nu \boldsymbol{u}_{\parallel})\, \mathrm{d}a = -\int_{\mathcal{S}} u_\nu \nabla_{\mathrm{s}} H \cdot \boldsymbol{u}_{\parallel}\, \mathrm{d}a + \int_{\mathcal{C}} u_\nu (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel} \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s.$$

Thus, (C.12) becomes

$$E_6 = \frac{\gamma}{2}\int_{\mathcal{S}} \left[|\nabla_{\mathrm{s}} u_\nu|^2 + (2K - H^2)u_\nu^2\right] \mathrm{d}a + \frac{\gamma}{2}\int_{\mathcal{S}} u_\nu \nabla_{\mathrm{s}} H \cdot \boldsymbol{u}_{\parallel}\, \mathrm{d}a$$
$$+ \frac{\gamma}{2}\int_{\mathcal{S}} H u_\nu \boldsymbol{u} \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s - \gamma \int_{\mathcal{S}} u_\nu (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_{\parallel} \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s$$
$$- \frac{\gamma}{2}\int_{\mathcal{C}} [(\nabla_{\mathrm{s}}\boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel} - (\mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}_{\parallel})\boldsymbol{u}_{\parallel}] \cdot \boldsymbol{\nu}_{\mathcal{S}}\, \mathrm{d}s.$$

Taking the surface-gradient of both sides of equation $(2.44)_1$ and recalling equation $(C.3)_1$ we obtain

$$E_3 + \frac{\gamma}{2}\int_{\mathcal{S}} u_\nu \nabla_{\mathrm{s}} H \cdot \boldsymbol{u}_{\parallel}\, \mathrm{d}a = 0$$

and so, also with the aid of (C.3) and (C.5), we transform (C.6) into (3.4).

Appendix D. Second variation of $\mathcal{F}_w$

It follows from (2.40) that:

$$\frac{1}{2}\delta^2\mathcal{F}_w[\mathcal{S}_*] = -\int_{\mathcal{S}_*}\left\{\left[-\frac{1}{2}Hw\boldsymbol{u}\cdot(\nabla_{\rm s}\boldsymbol{\nu}_*)\boldsymbol{u} + w\left(\frac{1}{2}|\boldsymbol{a}|^2 + I_2(\nabla_{\rm s}\boldsymbol{u})\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\big(\nabla_{\rm s}^2 w\big)\boldsymbol{u}\cdot\boldsymbol{u} + (\mathrm{div}_{\rm s}\,\boldsymbol{u})(\nabla_{\rm s}w)\cdot\boldsymbol{u}\right]\right\}\mathrm{d}a - \int_{\mathcal{C}} w\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s. \tag{D.1}$$

Let

$$E_7 := \int_{\mathcal{S}_*} wI_2(\nabla_{\rm s}\boldsymbol{u})\,\mathrm{d}a. \tag{D.2}$$

Since $\boldsymbol{u}$ is everywhere tangent to $\mathcal{S}_*$, by equation (A.7) applied with $\boldsymbol{g} = \boldsymbol{u}$, the integral in (D.2) can be recast as

$$E_7 = \int_{\mathcal{S}_*} w\left\{\frac{1}{2}\,\mathrm{div}_{\rm s}[(\mathrm{div}_{\rm s}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u}] - \frac{1}{2}|(\nabla_{\rm s}\boldsymbol{\nu}_*)\boldsymbol{u}|^2\right\}\mathrm{d}a.$$

By applying equation (A.8) on $\mathcal{S}_*$ with $\boldsymbol{g} = \boldsymbol{u}$, and by resorting to the surface-divergence theorem, one then obtains

$$E_7 = \frac{1}{2}\int_{\mathcal{C}} w[(\mathrm{div}_{\rm s}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{S}_*}\{wKu^2 - \nabla_{\rm s}w\cdot[(\mathrm{div}_{\rm s}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u}]\}\,\mathrm{d}a$$

and so (D.1) becomes

$$\delta^2\mathcal{F}_w[\mathcal{S}_*] = -\int_{\mathcal{S}_*}\left\{\frac{w}{2}[\boldsymbol{u}\cdot(\nabla_{\rm s}\boldsymbol{\nu}_*)^2\boldsymbol{u} - H\boldsymbol{u}\cdot(\nabla_{\rm s}\boldsymbol{\nu}_*)\boldsymbol{u} + Ku^2]\right.$$
$$\left. + \frac{1}{2}\big[\big(\nabla_{\rm s}^2 w\big)\boldsymbol{u}\cdot\boldsymbol{u} + \mathrm{div}_{\rm s}\,\boldsymbol{u}(\nabla_{\rm s}w)\cdot\boldsymbol{u} + \nabla_{\rm s}w\cdot(\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u}\big]\right\}\mathrm{d}a$$
$$-\int_{\mathcal{C}} w\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s - \frac{1}{2}\int_{\mathcal{C}} w[(\mathrm{div}_{\rm s}\,\boldsymbol{u})\boldsymbol{u} - (\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s.$$

By using the identity

$$\mathrm{div}_{\rm s}\{[(\nabla_{\rm s}w)\cdot\boldsymbol{u}]\boldsymbol{u}\} = \big(\nabla_{\rm s}^2 w\big)\boldsymbol{u}\cdot\boldsymbol{u} + \nabla_{\rm s}w\cdot(\nabla_{\rm s}\boldsymbol{u})\boldsymbol{u} + (\nabla_{\rm s}w)\cdot\boldsymbol{u}(\mathrm{div}_{\rm s}\,\boldsymbol{u})$$

and employing lemma 2.1 and the surface-divergence theorem, one reduces $\delta^2\mathcal{F}_w$ to the form in (3.5).

Appendix E. Second variation of $\mathcal{F}_\ell$

Note that, by $(2.26)_{1,2}$,

$$\boldsymbol{u}' = (u'_{t*} + u_{s*}\kappa_g^*)\boldsymbol{t}_* + (u'_{s*} - u_{t*}\kappa_g^*)\boldsymbol{\nu}_{\mathcal{S}_*} + (u_{s*}\tau_g^* - u_{t*}\kappa_n^*)\boldsymbol{\nu}_* \tag{E.1}$$

where a prime denotes differentiation with respect to the arc-length s and $\boldsymbol{t}_* = -\boldsymbol{t}$.

Then we write

$$\big(\nabla_{\rm s}^2\tau\big)\boldsymbol{u}\cdot\boldsymbol{u} = u_{t*}^2\big(\nabla_{\rm s}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{t}_* + u_{s*}^2\big(\nabla_{\rm s}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + u_{t*}u_{s*}\big[\big(\nabla_{\rm s}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{\nu}_{\mathcal{S}_*} + \big(\nabla_{\rm s}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t}_*\big] \tag{E.2}$$

and we insert this equation along with (E.1) in (2.43), thus arriving at

$$\begin{aligned}\frac{1}{2}\delta^2\mathcal{F}_\ell[\mathcal{C}] = \int_{\mathcal{C}} \Big\{ &(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})(\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{v}) - \tau\left[\kappa_g^*\boldsymbol{v}\cdot\boldsymbol{\nu}_{\mathcal{S}_*} - \frac{1}{2}\kappa_n^*\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}\right]\\ &+\frac{1}{2}\tau[(u_{s*}'-u_{t*}\kappa_g^*)^2+(u_{s*}\tau_g^*-u_{t*}\kappa_n^*)^2]+\frac{1}{2}\left[u_{t*}^2\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{t}_*\right.\\ &\left.+u_{s*}^2\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+u_{t*}u_{s*}\big[\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t}_*\big]\right]\\ &-(u_{t*}\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*+u_{s*}\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})(u_{t*}'+\kappa_g^*u_{s*})\Big\}\,\mathrm{d}s\end{aligned}\tag{E.3}$$

where also $(2.27)_2$ has been used. Moreover, since, again by $(2.26)_1$ and the identity $\boldsymbol{t}_*' = -\frac{\mathrm{d}\boldsymbol{t}_*}{\mathrm{d}s_*}$,

$$\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{t}_* = -(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*)' + \nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*' = -(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*)' - \kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}$$

and

$$\tfrac{1}{2}u_{t*}^2(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*)' + u_{t*}u_{t*}'\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_* = \tfrac{1}{2}[u_{t*}^2\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*]'$$

the integral of which along $\mathcal{C}$ is zero, we give $\delta^2\mathcal{F}_\ell$ the expression in (3.6).

Appendix F. Line integrals

The functional $\mathcal{F}_{\mathcal{C}}^{(2)}$ is defined by

$$\begin{aligned}\frac{1}{2}\mathcal{F}_{\mathcal{C}}^{(2)}[\boldsymbol{u}] := &\frac{\gamma}{2}\int_{\mathcal{C}} Hu_\nu\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s - \gamma\int_{\mathcal{C}} u_\nu(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{u}_\parallel\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s - \frac{\gamma}{2}\int_{\mathcal{C}}\sin\vartheta_c\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}\,\mathrm{d}s\\ &-\frac{\gamma}{2}\int_{\mathcal{C}}[(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel-(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel]\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s - \frac{\gamma}{2}\int_{\mathcal{C}}[(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}-(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s\\ &+\int_{\mathcal{C}}\frac{w}{2}[(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}-(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s - \frac{1}{2}\int_{\mathcal{C}}(\nabla_{\mathrm{s}}w)\cdot\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{\nu}_{\mathcal{S}_*})\,\mathrm{d}s\\ &+\int_{\mathcal{C}}\left\{\left[\frac{\tau}{2}\kappa_n^*\boldsymbol{u}\cdot(\nabla_{\mathrm{s}}\boldsymbol{\nu}_*)\boldsymbol{u}\right]+\frac{\tau}{2}[(u_{s*}'-u_{t*}\kappa_g^*)^2+(u_{s*}\tau_g^*-u_{t*}\kappa_n^*)^2]\right.\\ &-\frac{1}{2}u_{t*}^2\kappa_g^*(\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*})+\frac{1}{2}u_{s*}^2\big[\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{\nu}_{\mathcal{S}_*}-2\kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\big]-u_{t*}'u_{s*}\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\\ &\left.+\frac{1}{2}u_{s*}u_{t*}\big[\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{t}_*\cdot\boldsymbol{\nu}_{\mathcal{S}_*}+\big(\nabla_{\mathrm{s}}^2\tau\big)\boldsymbol{\nu}_{\mathcal{S}_*}\cdot\boldsymbol{t}_*-2\kappa_g^*\nabla_{\mathrm{s}}\tau\cdot\boldsymbol{t}_*\big]\right\}\mathrm{d}s.\end{aligned}\tag{F.1}$$

We first put together the fourth, the fifth and the sixth integral on the right-hand side of equation (F.1):

$$L_1 := -\frac{\gamma}{2}\int_{\mathcal{C}}[(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel-(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel]\cdot\boldsymbol{\nu}_{\mathcal{S}}\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{C}}(w-\gamma)[(\nabla_{\mathrm{s}}\boldsymbol{u})\boldsymbol{u}-(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u})\boldsymbol{u}]\cdot\boldsymbol{\nu}_{\mathcal{S}_*}\,\mathrm{d}s.$$

We then decompose $\boldsymbol{u}_\parallel$ in the local basis $\{\boldsymbol{t},\boldsymbol{\nu}_{\mathcal{S}}\}$ along $\mathcal{C}$ as

$$\boldsymbol{u}_\parallel = u_t\boldsymbol{t}+u_s\boldsymbol{\nu}_{\mathcal{S}}\tag{F.2}$$

whence we obtain that

$$\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel = u_t\nabla_{\mathrm{s}}\boldsymbol{t}+\boldsymbol{t}\otimes\nabla_{\mathrm{s}}u_t+u_s\nabla_{\mathrm{s}}\boldsymbol{\nu}_{\mathcal{S}}+\boldsymbol{\nu}_{\mathcal{S}}\otimes\nabla_{\mathrm{s}}u_s\tag{F.3}$$

and so

$$\begin{aligned}(\nabla_{\mathrm{s}}\boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel = &u_t^2(\nabla_{\mathrm{s}}\boldsymbol{t})\boldsymbol{t}+u_t(\nabla_{\mathrm{s}}u_t\cdot\boldsymbol{t})\boldsymbol{t}+u_su_t(\nabla_{\mathrm{s}}\boldsymbol{\nu}_{\mathcal{S}})\boldsymbol{t}+u_t(\nabla_{\mathrm{s}}u_s\cdot\boldsymbol{t})\boldsymbol{\nu}_{\mathcal{S}}\\ &+u_su_t(\nabla_{\mathrm{s}}\boldsymbol{t})\boldsymbol{\nu}_{\mathcal{S}}+u_s(\nabla_{\mathrm{s}}u_t\cdot\boldsymbol{\nu}_{\mathcal{S}})\boldsymbol{t}+u_s^2(\nabla_{\mathrm{s}}\boldsymbol{\nu}_{\mathcal{S}})\boldsymbol{\nu}_{\mathcal{S}}+u_s(\nabla_{\mathrm{s}}u_s\cdot\boldsymbol{\nu}_{\mathcal{S}})\boldsymbol{\nu}_{\mathcal{S}}.\end{aligned}\tag{F.4}$$

By direct computation starting from (2.1) (see [38]) one obtains

$$\begin{cases} \nabla_{\mathrm{s}} \boldsymbol{t} = \kappa_g \boldsymbol{\nu}_{\mathcal{S}} \otimes \boldsymbol{t} + \kappa_n \boldsymbol{\nu} \otimes \boldsymbol{t} + \kappa_{g\perp} \boldsymbol{\nu}_{\mathcal{S}} \otimes \boldsymbol{\nu}_{\mathcal{S}} - \tau_g \boldsymbol{\nu} \otimes \boldsymbol{\nu}_{\mathcal{S}} \\ \nabla_{\mathrm{s}} \boldsymbol{\nu}_{\mathcal{S}} = -\kappa_g \boldsymbol{t} \otimes \boldsymbol{t} + \kappa_{n\perp} \boldsymbol{\nu} \otimes \boldsymbol{\nu}_{\mathcal{S}} - \kappa_{g\perp} \boldsymbol{t} \otimes \boldsymbol{\nu}_{\mathcal{S}} - \tau_g \boldsymbol{\nu} \otimes \boldsymbol{t} \\ \nabla_{\mathrm{s}} \boldsymbol{\nu} = -\kappa_n \boldsymbol{t} \otimes \boldsymbol{t} - \kappa_{n\perp} \boldsymbol{\nu}_{\mathcal{S}} \otimes \boldsymbol{\nu}_{\mathcal{S}} + \tau_g (\boldsymbol{\nu}_{\mathcal{S}} \otimes \boldsymbol{t} + \boldsymbol{t} \otimes \boldsymbol{\nu}_{\mathcal{S}}) \end{cases} \tag{F.5}$$

where, at any point of $\mathcal{C}$, $\kappa_{g\perp}$ and $\kappa_{n\perp}$ are the geodesic and normal curvatures of a curve on $\mathcal{S}$ orthogonal to $\mathcal{C}$ that inherits the orientation induced by $\boldsymbol{\nu}_{\mathcal{S}}$.[1] In what follows, we will also need a similar set of equations, obtained from (2.26):

$$\begin{cases} \nabla_{\mathrm{s}} \boldsymbol{t}_* = \kappa_g^* \boldsymbol{\nu}_{\mathcal{S}_*} \otimes \boldsymbol{t}_* + \kappa_n^* \boldsymbol{\nu}_* \otimes \boldsymbol{t}_* + \kappa_{g\perp}^* \boldsymbol{\nu}_{\mathcal{S}_*} \otimes \boldsymbol{\nu}_{\mathcal{S}_*} - \tau_g^* \boldsymbol{\nu}_* \otimes \boldsymbol{\nu}_{\mathcal{S}_*} \\ \nabla_{\mathrm{s}} \boldsymbol{\nu}_{\mathcal{S}_*} = -\kappa_g^* \boldsymbol{t}_* \otimes \boldsymbol{t}_* + \kappa_{n\perp}^* \boldsymbol{\nu}_* \otimes \boldsymbol{\nu}_{\mathcal{S}_*} - \kappa_{g\perp}^* \boldsymbol{t}_* \otimes \boldsymbol{\nu}_{\mathcal{S}_*} - \tau_g^* \boldsymbol{\nu}_* \otimes \boldsymbol{t}_* \\ \nabla_{\mathrm{s}} \boldsymbol{\nu}_* = -\kappa_n^* \boldsymbol{t}_* \otimes \boldsymbol{t}_* - \kappa_{n\perp}^* \boldsymbol{\nu}_{\mathcal{S}_*} \otimes \boldsymbol{\nu}_{\mathcal{S}_*} + \tau_g^* (\boldsymbol{\nu}_{\mathcal{S}_*} \otimes \boldsymbol{t}_* + \boldsymbol{t}_* \otimes \boldsymbol{\nu}_{\mathcal{S}_*}) \end{cases} \tag{F.6}$$

where now all quantities are referred to $\mathcal{C}$ seen as a curve on $\mathcal{S}_*$ and to auxiliary curves on $\mathcal{S}_*$. Inserting (F.5) into (F.3) and (F.4) we arrive at

$$[(\nabla_{\mathrm{s}} \boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel - (\mathrm{div}_{\mathrm{s}}\, \boldsymbol{u}_\parallel)\boldsymbol{u}_\parallel] \cdot \boldsymbol{\nu}_{\mathcal{S}} = \kappa_g \big(u_t^2 + u_s^2\big) + u_t (\nabla_{\mathrm{s}} u_s \cdot \boldsymbol{t}) - u_s (\nabla_{\mathrm{s}} u_t \cdot \boldsymbol{t}).$$

Likewise, also applying (3.7) and (F.6) we conclude that

$$\begin{aligned} L_1 = \int_{\mathcal{C}} \Big\{ & -\frac{\gamma}{2} \big[\kappa_g \big(u_t^2 + u_s^2\big) + u_t (\nabla_{\mathrm{s}} u_s \cdot \boldsymbol{t}) - u_s (\nabla_{\mathrm{s}} u_t \cdot \boldsymbol{t})\big] \\ & + \frac{1}{2}(w - \gamma) \big[\kappa_g^* \big(u_{t*}^2 + u_{s*}^2\big) + u_{t*} (\nabla_{\mathrm{s}} u_{s*} \cdot \boldsymbol{t}_*) - u_{s*} (\nabla_{\mathrm{s}} u_{t*} \cdot \boldsymbol{t}_*)\big] \Big\} \, \mathrm{d}s. \end{aligned} \tag{F.7}$$

Moreover, by recalling that $\boldsymbol{t} = -\boldsymbol{t}_*$ and by using equations $(2.27)_1$ and (2.25), we readily see that

$$u_{t*} = -u_t \qquad u_s = \cos\vartheta_c u_{s*} \qquad \text{and} \qquad u_\nu = \sin\vartheta_c u_{s*} \tag{F.8}$$

and so we can express the right-hand side of (F.7) only in terms of u_t and u_{s*}:

$$\begin{aligned} L_1 = \int_{\mathcal{C}} & \frac{1}{2} u_t^2 [-\kappa_g \gamma + \kappa_g^* (w - \gamma)] + \frac{1}{2} u_{s*}^2 [-\gamma \kappa_g \cos^2\vartheta_c + \kappa_g^* (w - \gamma)] \\ & - \frac{1}{2} u_t [\gamma (\cos\vartheta_c u_{s*})' - (w - \gamma) u_{s*}'] + \frac{1}{2} u_{s*} u_t' [\gamma \cos\vartheta_c - (w - \gamma)] \, \mathrm{d}s. \end{aligned} \tag{F.9}$$

Similarly, let L_2 be the sum of the first two integrals in (F.1):

$$L_2 := \frac{\gamma}{2} \int_{\mathcal{C}} H u_\nu \boldsymbol{u} \cdot \boldsymbol{\nu}_{\mathcal{S}} \, \mathrm{d}s - \gamma \int_{\mathcal{C}} u_\nu (\nabla_{\mathrm{s}} \boldsymbol{\nu}) \boldsymbol{u}_\parallel \cdot \boldsymbol{\nu}_{\mathcal{S}} \, \mathrm{d}s.$$

Using $(F.5)_3$ and (F.2) one finds that

$$L_2 = \int_{\mathcal{C}} \gamma \left[u_\nu u_s \left(\frac{H}{2} + \kappa_{n\perp} \right) - u_t u_\nu \tau_g \right] \mathrm{d}s. \tag{F.10}$$

To simplify the integrands of both the third and the eighth integrals in (F.1), we compute

$$\boldsymbol{u} \cdot (\nabla_{\mathrm{s}} \boldsymbol{\nu}_*) \boldsymbol{u} = 2\tau_g^* u_{t*} u_{s*} - u_{t*}^2 \kappa_n^* - u_{s*}^2 \kappa_{n\perp}^* = -\big(2\tau_g^* u_t u_{s*} + u_t^2 \kappa_n^* + u_{s*}^2 \kappa_{n\perp}^*\big) \tag{F.11}$$

where both $(F.5)_3$ and $(F.8)_1$ have been used. Finally, we note that, by $(F.8)_1$, in the seventh integral of (F.1)

$$\nabla_{\mathrm{s}} w \cdot \boldsymbol{u} = w' u_t + u_{s*} (\nabla_{\mathrm{s}} w \cdot \boldsymbol{\nu}_{\mathcal{S}_*}). \tag{F.12}$$

[1] This curve is not unique: only its normal curvature $\kappa_{n\perp}$ is prescribed at the point where it crosses $\mathcal{C}$, while $\kappa_{g\perp}$ is not. However, such an indeterminacy does not affect our analysis, as our final result does not depend on $\kappa_{g\perp}$.

Hence, inserting equations (F.9), (F.10), (F.11) and (F.12) into (F.1) and making repeated use of (F.8), we find that

$$\begin{aligned}\frac{1}{2}\mathcal{F}_{\mathcal{C}}^{(2)} = \int_{\mathcal{C}} \Bigg\{ & \gamma \left[u_{s*}^2 \sin\vartheta_c \cos\vartheta_c \left(\frac{H}{2} + \kappa_{n\perp} \right) - u_t u_{s*} \tau_g \sin\vartheta_c \right] \\ & + \frac{1}{2} u_t^2 [-\kappa_g \gamma + \kappa_g^* (w-\gamma)] + \frac{1}{2} u_{s*}^2 [-\gamma \kappa_g \cos^2\vartheta_c + \kappa_g^* (w-\gamma)] \\ & - \frac{1}{2} u_t [\gamma (\cos\vartheta_c u_{s*})' - (w-\gamma) u_{s*}'] + \frac{1}{2} u_{s*} u_t' [\gamma \cos\vartheta_c - (w-\gamma)] \\ & + \frac{\gamma}{2} \sin\vartheta_c \big[2\tau_g^* u_t u_{s*} + u_t^2 \kappa_n^* + u_{s*}^2 \kappa_{n\perp}^* \big] - \frac{1}{2} [w' u_t + (\nabla_{\mathrm{s}} w \cdot \boldsymbol{\nu}_{\mathcal{S}_*}) u_{s*}] u_{s*} \\ & - \frac{\tau}{2} \kappa_n^* \big(2\tau_g^* u_t u_{s*} + u_t^2 \kappa_n^* + u_{s*}^2 \kappa_{n\perp}^* \big) + \frac{\tau}{2} \big[\kappa_g^{*2} u_t^2 + 2\kappa_g^* u_t u_{s*}' + u_{s*}'^2 \\ & + \kappa_n^{*2} u_t^2 + \tau_g^{*2} u_{s*}^2 + 2\kappa_n^* \tau_g^* u_t u_{s*} \big] - \frac{1}{2} u_t^2 \kappa_g^* (\nabla_{\mathrm{s}} \tau \cdot \boldsymbol{\nu}_{\mathcal{S}_*}) \\ & + \frac{1}{2} u_{s*}^2 \big[(\nabla_{\mathrm{s}}^2 \tau) \boldsymbol{\nu}_{\mathcal{S}_*} \cdot \boldsymbol{\nu}_{\mathcal{S}_*} - 2\kappa_g^* \nabla_{\mathrm{s}} \tau \cdot \boldsymbol{\nu}_{\mathcal{S}_*} \big] - u_{t*}' u_{s*} \nabla_{\mathrm{s}} \tau \cdot \boldsymbol{\nu}_{\mathcal{S}_*} \\ & - \frac{1}{2} u_{s*} u_t \big[(\nabla_{\mathrm{s}}^2 \tau) \boldsymbol{t}_* \cdot \boldsymbol{\nu}_{\mathcal{S}_*} + (\nabla_{\mathrm{s}}^2 \tau) \boldsymbol{\nu}_{\mathcal{S}_*} \cdot \boldsymbol{t}_* - 2\kappa_g^* \nabla_{\mathrm{s}} \tau \cdot \boldsymbol{t}_* \big] \Bigg\} \mathrm{d}s. \end{aligned} \tag{F.13}$$

Using the identity

$$(\cos\vartheta_c u_{s*})' = (\cos\vartheta_c)' u_{s*} + \cos\vartheta_c u_{s*}'$$

and collecting similar terms in (F.13), we easily arrive at (3.8).

Appendix G. Geodesic torsions

For completeness, we prove here equation (3.10). Let $\mathcal{S}$ and $\mathcal{S}_*$ be two surfaces that intersect each other along the closed curve $\mathcal{C}$. Let φ be the angle between the principal normal $\boldsymbol{n}$ to $\mathcal{C}$ and the outer normal vector $\boldsymbol{\nu}$ to $\mathcal{S}$, so that

$$\boldsymbol{n} = \cos\varphi \boldsymbol{\nu} - \sin\varphi \boldsymbol{\nu}_{\mathcal{S}} \tag{G.1}$$

where $\boldsymbol{\nu}_{\mathcal{S}}$ is the conormal vector to $\mathcal{S}$ along $\mathcal{C}$ defined in section 2. The binormal unit vector $\boldsymbol{b}$ to $\mathcal{C}$ is defined as $\boldsymbol{b} := \boldsymbol{t} \times \boldsymbol{n}$. Since $\boldsymbol{\nu}_{\mathcal{S}} = \boldsymbol{\nu} \times \boldsymbol{t}$, we can decompose $\boldsymbol{b}$ in the Darboux trihedron associated with $\mathcal{C}$ regarded as a curve on $\mathcal{S}$, thus obtaining

$$\boldsymbol{b} = -\cos\varphi \boldsymbol{\nu}_{\mathcal{S}} - \sin\varphi \boldsymbol{\nu}. \tag{G.2}$$

We differentiate the identity $\boldsymbol{n} \cdot \boldsymbol{\nu} = \cos\varphi$ with respect to s, using $(2.1)_3$ and the Frénet–Serret equation for space curves

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = -\sigma \boldsymbol{t} - \tau_{\mathcal{C}} \boldsymbol{b}$$

where $\tau_{\mathcal{C}}$ is the torsion of $\mathcal{C}$: we thus arrive at

$$\sin\varphi \frac{\mathrm{d}\varphi}{\mathrm{d}s} = (\tau_g - \tau_{\mathcal{C}}) \sin\varphi$$

whence it follows that

$$-\frac{\mathrm{d}\varphi}{\mathrm{d}s} = (\tau_{\mathcal{C}} - \tau_g). \tag{G.3}$$

Similarly, we decompose $\boldsymbol{n}$ in the Darboux trihedron associated with $\mathcal{C}$ regarded as a curve on $\mathcal{S}_*$:

$$\boldsymbol{n} = \cos\varphi_* \boldsymbol{\nu}_* - \sin\varphi_* \boldsymbol{\nu}_{\mathcal{S}_*}.$$

We reason again as above, observing however that the orientation of $\mathcal{C}$ must be changed to preserve the same orientation of the Darboux trihedron. Thus, (G.3) is replaced by

$$-\frac{\mathrm{d}\varphi_*}{\mathrm{d}s} = \frac{\mathrm{d}\varphi_*}{\mathrm{d}s_*} = (\tau_{\mathcal{C}} - \tau_g^*). \tag{G.4}$$

Finally, since $\varphi + \varphi_* - \vartheta_c = \pi$, we obtain from (G.3) and (G.4) that

$$-\frac{\mathrm{d}\vartheta_c}{\mathrm{d}s} = \tau_g^* - \tau_g$$

which is the desired result (3.10).

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